

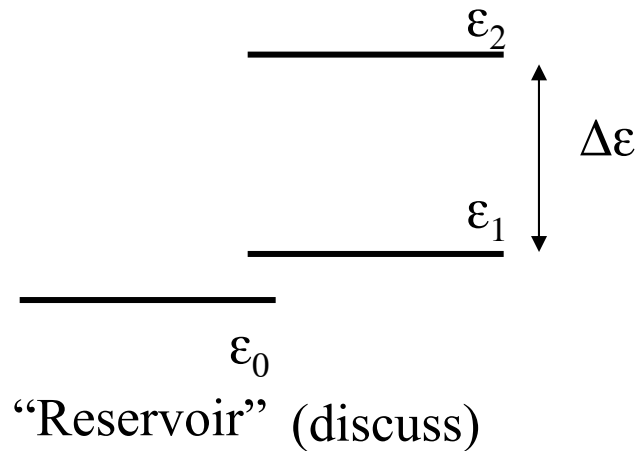
Lecture 6: Three-level rate equations

- Recall: No two-level laser (discuss)
- Three level laser: Let's try

(Note: Some audio missing because I went back in the presentation during the lecture.)



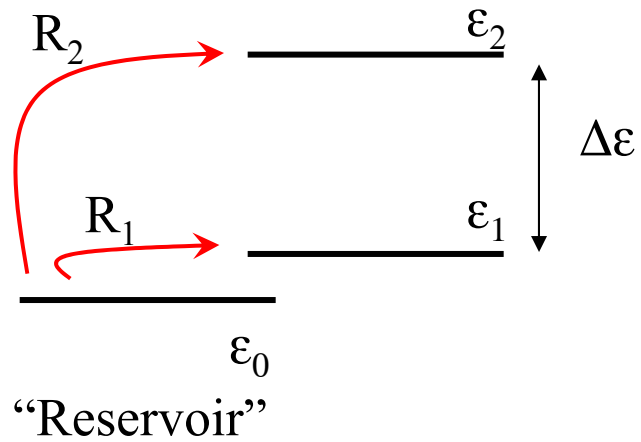
Three-level rate equations:



We want to have a set of equations to describe the populations of atoms in each state as a function of time and intensity.



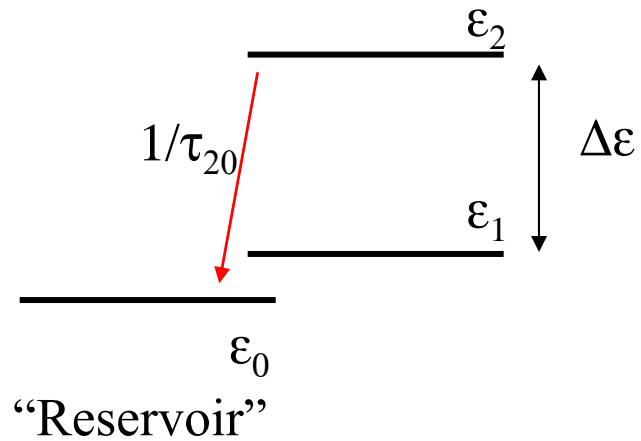
Three-level rate equations:



R_1 and R_2 are the external "pump" rates.



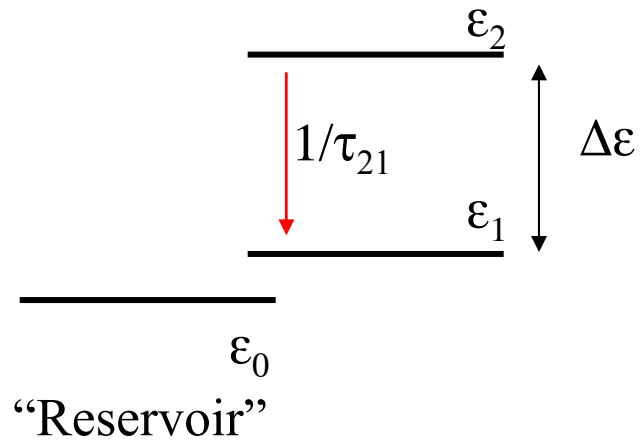
Three-level rate equations:



$1/\tau_{20}$ is spontaneous emission rate $2 \rightarrow 0$



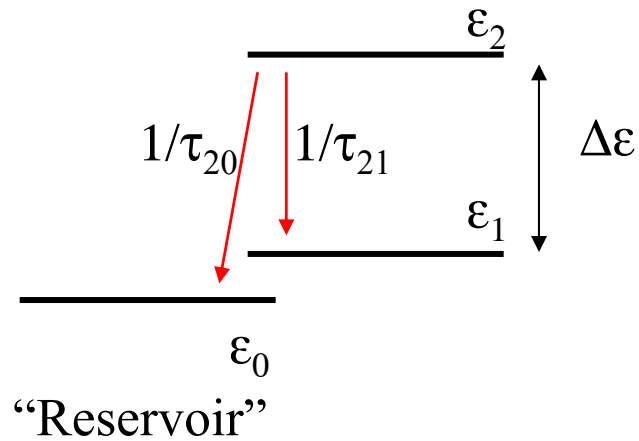
Three-level rate equations:



$1/\tau_{21}$ is spontaneous emission rate 2- \rightarrow 1



Three-level rate equations:



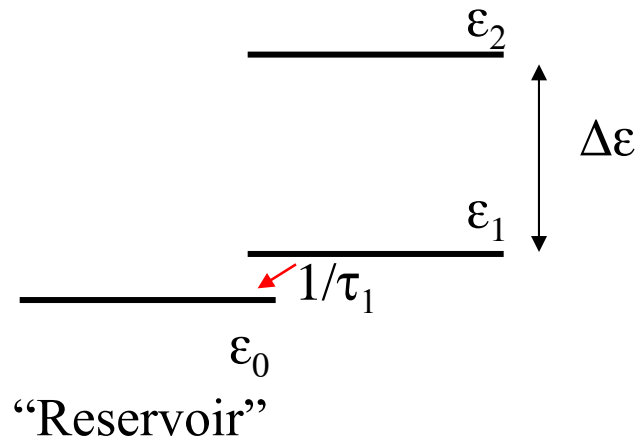
$1/\tau_{20}$ is spontaneous emission rate 2- \rightarrow 0

$1/\tau_{21}$ is spontaneous emission rate 2- \rightarrow 1

$1/\tau_2 = 1/\tau_{21} + 1/\tau_{20}$ spont em rate 2- \rightarrow (anything)



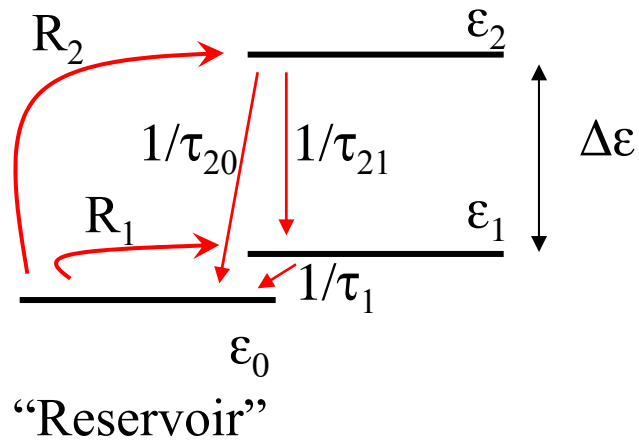
Three-level rate equations:



$1/\tau_1$ is spontaneous emission rate $1 \rightarrow 0$



Three-level rate equations:



R_1 and R_2 are the external "pump" rates.

$1/\tau_{20}$ is spontaneous emission rate $2 \rightarrow 0$

$1/\tau_{21}$ is spontaneous emission rate $2 \rightarrow 1$

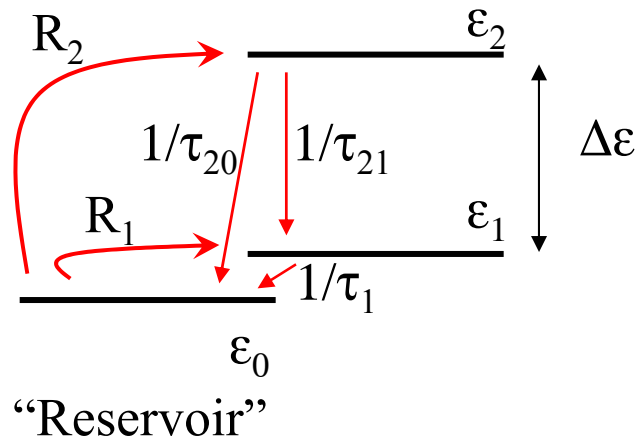
$1/\tau_2 = 1/\tau_{21} + 1/\tau_{20}$ spont em rate $2 \rightarrow$ (anything)

$1/\tau_1$ is spontaneous emission rate $1 \rightarrow 0$

Also, absorption $2 \rightarrow 1$ and stimulated emission!



Three-level rate equations:



R_1 and R_2 are the external “pump” rates.

$1/\tau_{20}$ is spontaneous emission rate $2 \rightarrow 0$

$1/\tau_{21}$ is spontaneous emission rate $2 \rightarrow 1$

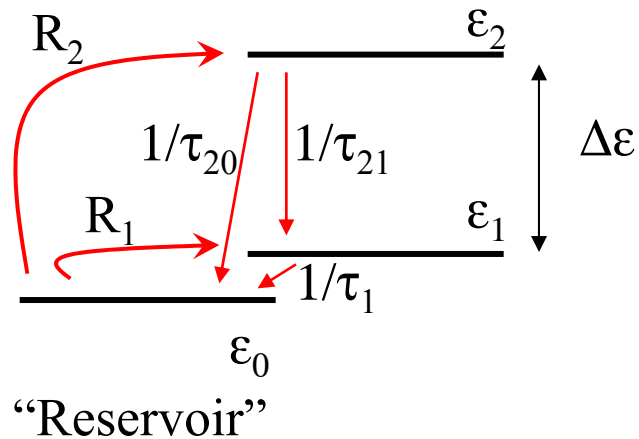
$1/\tau_2 = 1/\tau_{21} + 1/\tau_{20}$ spont em rate $2 \rightarrow$ (anything)

$1/\tau_1$ is spontaneous emission rate $1 \rightarrow 0$

Goal: Use R_2 to achieve steady-state population inversion.



Three-level rate equations:



R_1 and R_2 are the external “pump” rates.

$1/\tau_{20}$ is spontaneous emission rate $2 \rightarrow 0$

$1/\tau_{21}$ is spontaneous emission rate $2 \rightarrow 1$

$1/\tau_2 = 1/\tau_{21} + 1/\tau_{20}$ spont em rate $2 \rightarrow$ (anything)

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Goal: Use R_2 to achieve steady-state population inversion.

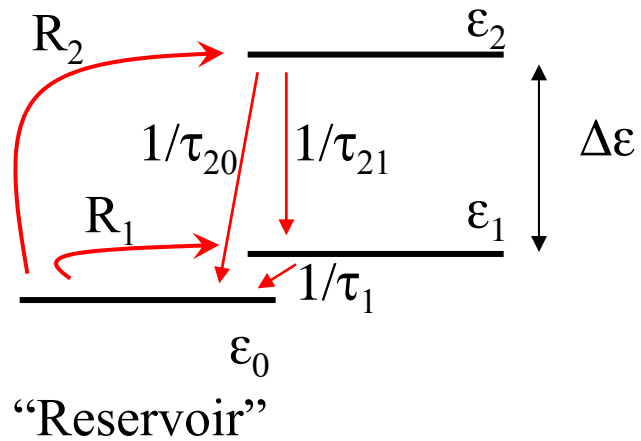
Guess:

Want lifetime in state 2 very long,

Want lifetime in state 1 very short.



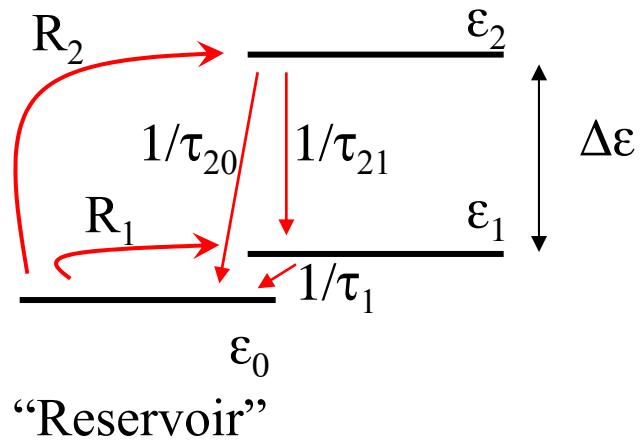
Three-level rate equations:



$$\frac{dN_2}{dt} = R_2(t)$$



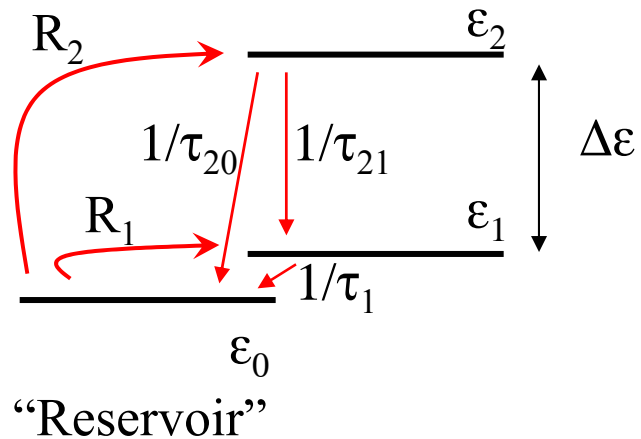
Three-level rate equations:



$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2} \right) N_2$$



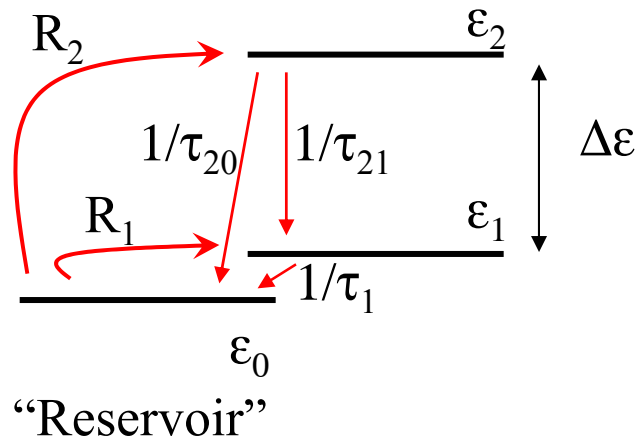
Three-level rate equations:



$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2} \right) N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$



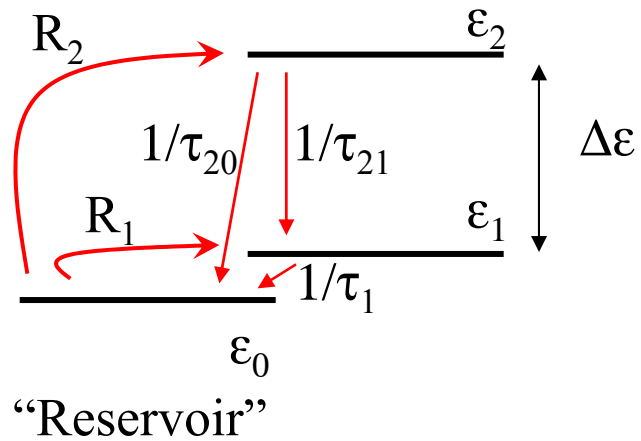
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$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2} \right) N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$



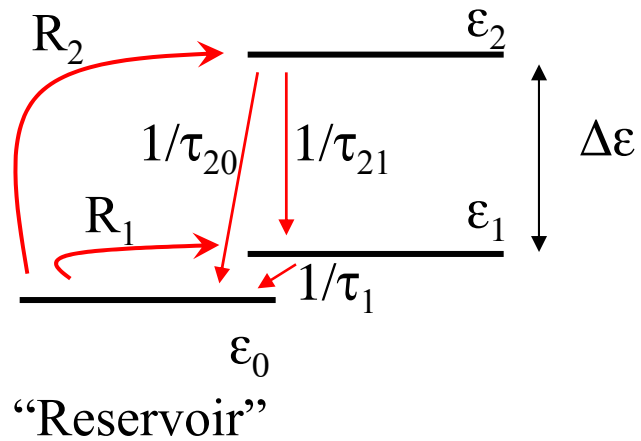
Three-level rate equations:



$$\frac{dN_1}{dt} = R_1(t)$$



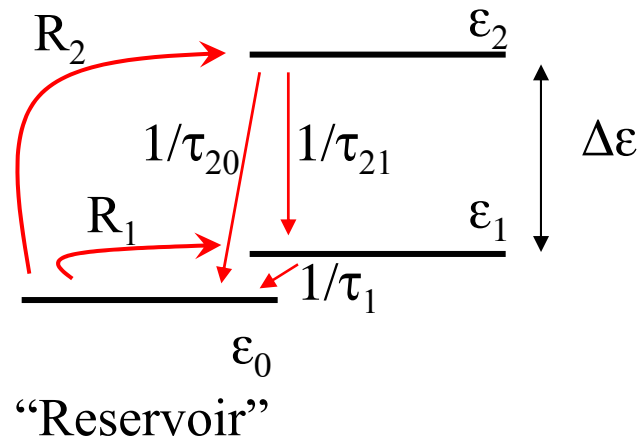
Three-level rate equations:



$$\frac{dN_1}{dt} = R_1(t) - \left(\frac{1}{\tau_{21}} \right) N_2$$



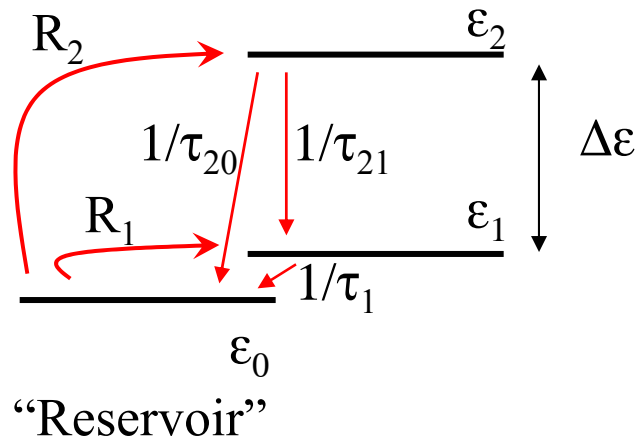
Three-level rate equations:



$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}} \right) N_2 + \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$



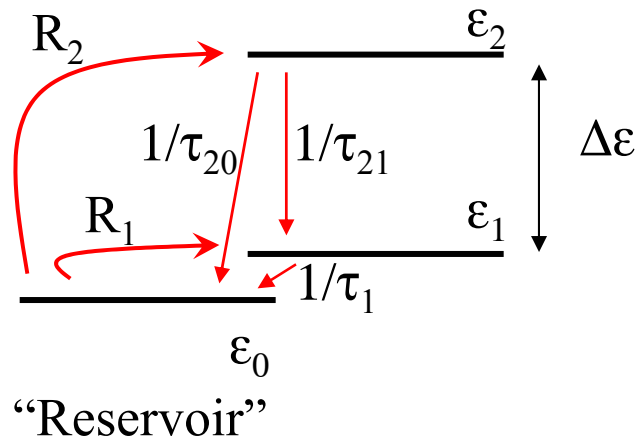
Three-level rate equations:



$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$



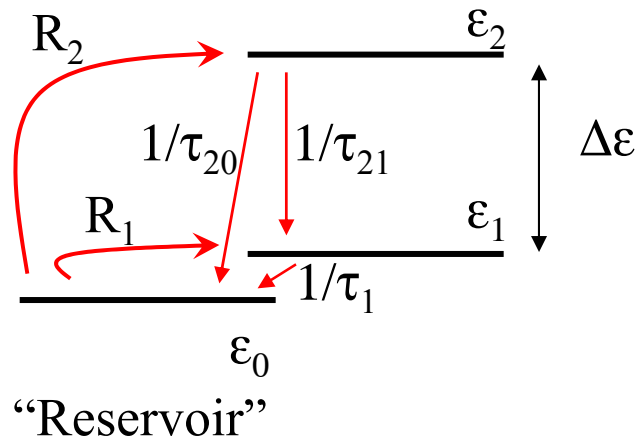
Three-level rate equations:



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Three-level rate equations:



$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)]$$

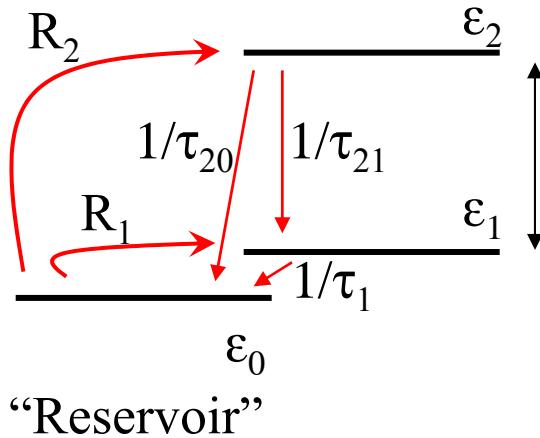
$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

Four cases to consider:

- 1) $I = 0$
- 2) I finite, $\tau_1 = 0$
- 3) I is pulsed.
- 4) General conditions for steady state.



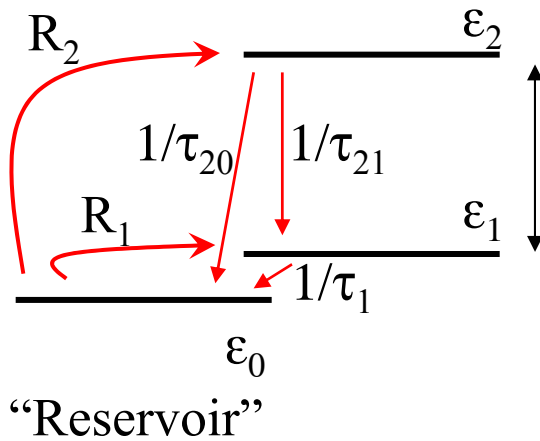
Case One: $I=0$



$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)]$$



Case One: $I=0$

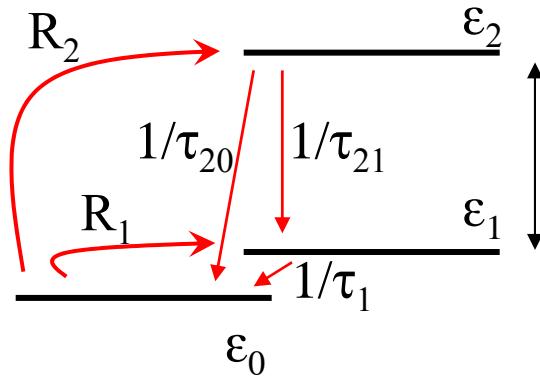


$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$

0



Case One: I=0



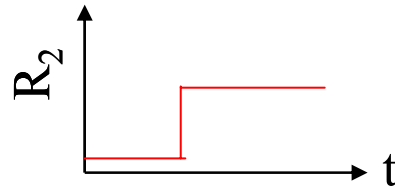
“Reservoir”

$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$

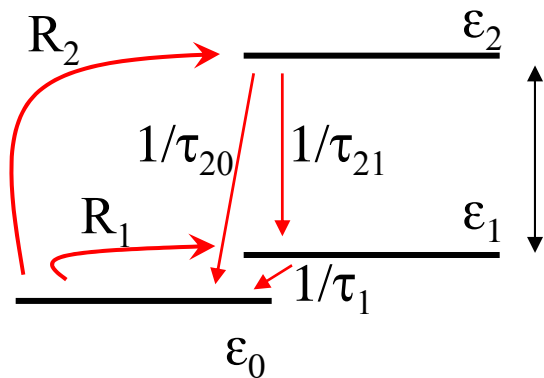
0

$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2$$

Let: $R_2(t) = R_{20}u(t)$



Case One: I=0

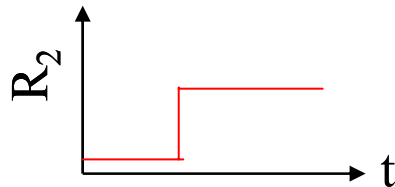


“Reservoir”

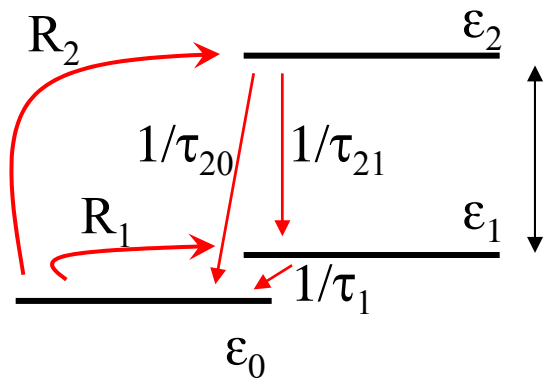
$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$

$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2$$

Let: $R_2(t) = R_{20}u(t)$ $N_2(0) = 0$



Case One: $I=0$

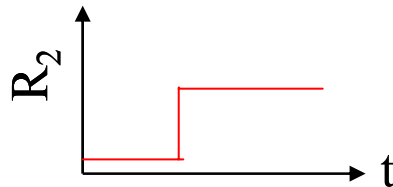


“Reservoir”

$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$

$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2$$

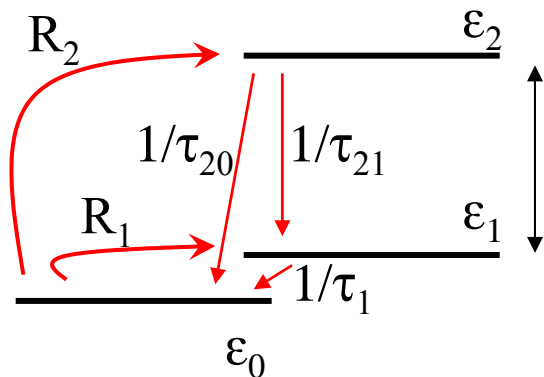
Let: $R_2(t) = R_{20}u(t)$ $N_2(0) = 0$



$$N_2(t) = \int \frac{dN_2}{dt} dt = R_{20}\tau_2 \left(1 - e^{-t/\tau_2}\right)$$



Case One: I=0

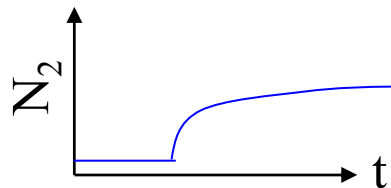
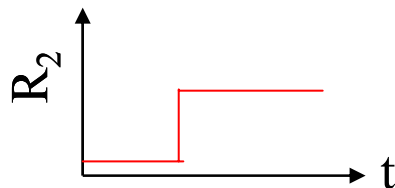


“Reservoir”

$$\Delta\epsilon \quad \frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$

$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2$$

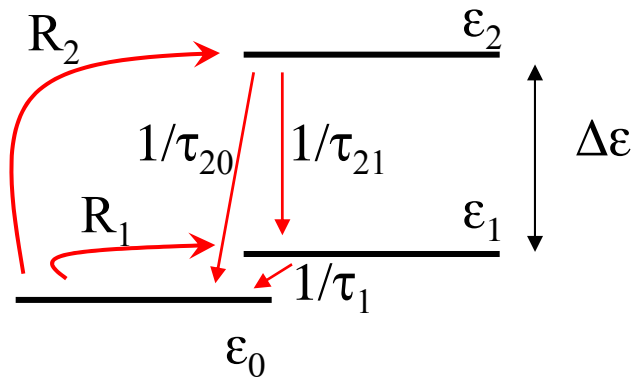
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Case One: $I=0$

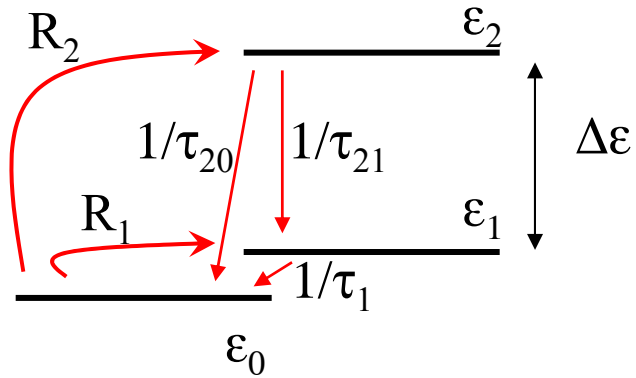


$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

"Reservoir"



Case One: $I=0$



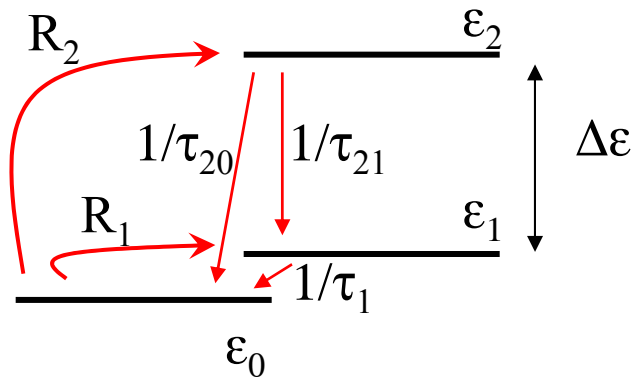
$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu)(N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

Red arrows point from the terms $R_1(t)$ and $\frac{I_\nu}{h\nu}[\sigma(\nu)(N_2 - N_1)]$ to the value 0, indicating they are zero in this case.

“Reservoir”



Case One: $I=0$



“Reservoir”

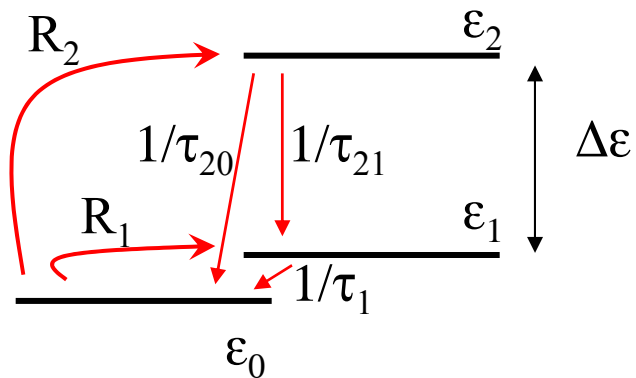
$$\frac{dN_1}{dt} = \cancel{R_1(t)} + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu)(N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

0 0

$$\frac{dN_1}{dt} = +\left(\frac{1}{\tau_{21}}\right)N_2 - \left(\frac{1}{\tau_1}\right)N_1$$



Case One: $I=0$



“Reservoir”

$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_v}{h\nu}[\sigma(\nu)(N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

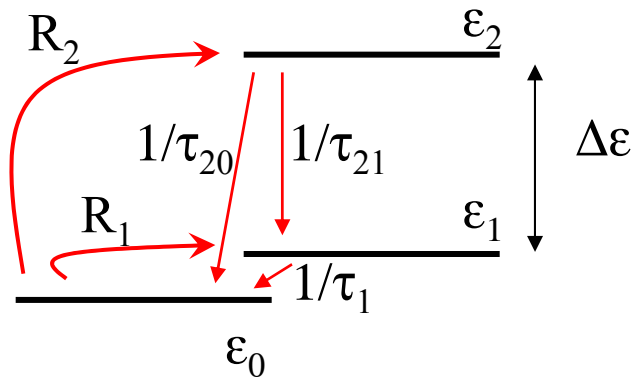
↙ ↘

$$\frac{dN_1}{dt} = +\left(\frac{1}{\tau_{21}}\right)N_2 - \left(\frac{1}{\tau_1}\right)N_1$$

$$\frac{dN_1}{dt} + \left(\frac{1}{\tau_1}\right)N_1 = +\left(\frac{1}{\tau_{21}}\right)N_2$$



Case One: I=0



“Reservoir”

$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu)(N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

↙ ↘

0 0

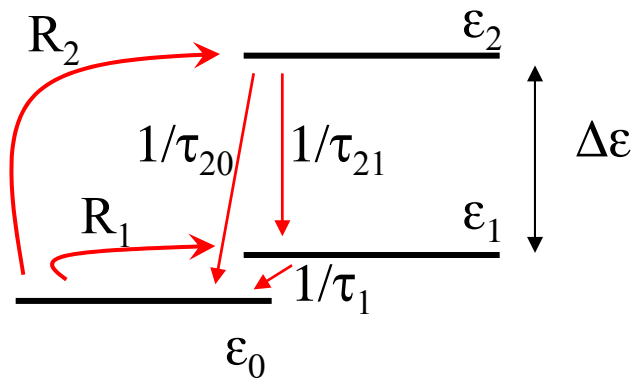
$$\frac{dN_1}{dt} = +\left(\frac{1}{\tau_{21}}\right)N_2 - \left(\frac{1}{\tau_1}\right)N_1$$

$$\frac{dN_1}{dt} + \left(\frac{1}{\tau_1}\right)N_1 = +\left(\frac{1}{\tau_{21}}\right)N_2$$

$$\frac{dN_1}{dt} e^{t/\tau_1} + \left(\frac{1}{\tau_1}\right)N_1 e^{t/\tau_1} = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$



Case One: I=0



$$\frac{dN_1}{dt} = \cancel{R_1(t)} + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu)(N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

0
0

$$\frac{dN_1}{dt} = +\left(\frac{1}{\tau_{21}}\right)N_2 - \left(\frac{1}{\tau_1}\right)N_1$$

$$\frac{dN_1}{dt} + \left(\frac{1}{\tau_1}\right)N_1 = +\left(\frac{1}{\tau_{21}}\right)N_2$$

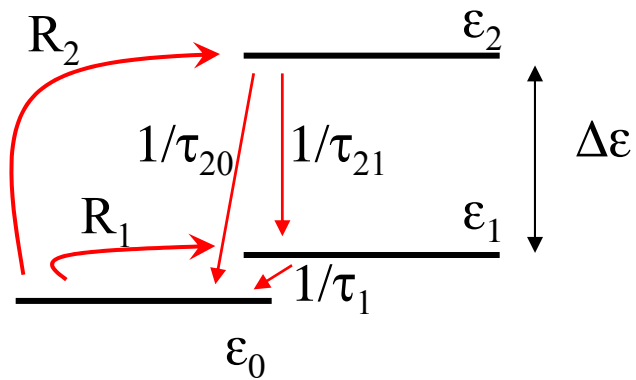
$$\frac{dN_1}{dt} e^{t/\tau_1} + \left(\frac{1}{\tau_1}\right)N_1 e^{t/\tau_1} = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

$$\frac{d}{dt} (N_1 e^{t/\tau_1}) = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

"Reservoir"



Case One: I=0



“Reservoir”

$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu)(N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

↙ ↘

$$\frac{dN_1}{dt} = +\left(\frac{1}{\tau_{21}}\right)N_2 - \left(\frac{1}{\tau_1}\right)N_1$$

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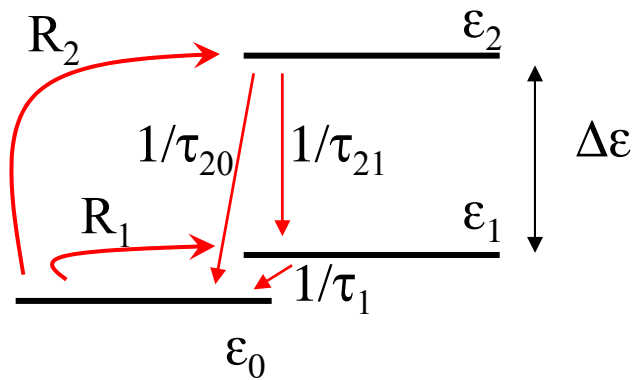
$$\frac{dN_1}{dt} e^{t/\tau_1} + \left(\frac{1}{\tau_1}\right)N_1 e^{t/\tau_1} = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

$$\frac{d}{dt} (N_1 e^{t/\tau_1}) = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

$$\int \frac{d}{dt} (N_1 e^{t/\tau_1}) = \int +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$



Case One: I=0



“Reservoir”

$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu)(N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

0
0

$$\frac{dN_1}{dt} = +\left(\frac{1}{\tau_{21}}\right)N_2 - \left(\frac{1}{\tau_1}\right)N_1$$

$$\frac{dN_1}{dt} + \left(\frac{1}{\tau_1}\right)N_1 = +\left(\frac{1}{\tau_{21}}\right)N_2$$

$$\frac{dN_1}{dt} e^{t/\tau_1} + \left(\frac{1}{\tau_1}\right)N_1 e^{t/\tau_1} = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

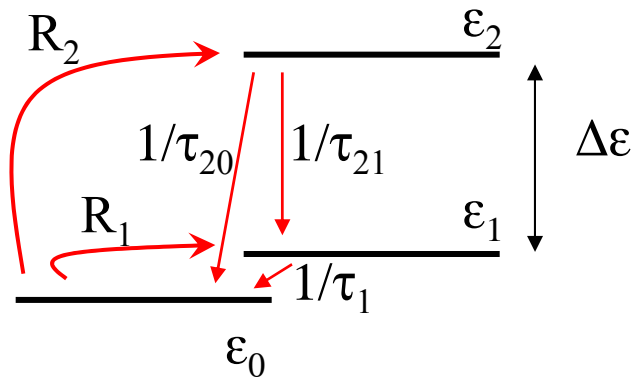
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$$\int \frac{d}{dt} (N_1 e^{t/\tau_1}) = \int +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

$$N_1 e^{t/\tau_1} = \int +\left(\frac{1}{\tau_{21}}\right)R_{20}\tau_2(1 - e^{-t/\tau_2})e^{t/\tau_1}$$



Case One: I=0



“Reservoir”

$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu)(N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

(Red arrows point from the terms $R_1(t)$ and $\frac{I_\nu}{h\nu}[\sigma(\nu)(N_2 - N_1)]$ to a red '0' below them.)

$$\frac{dN_1}{dt} = +\left(\frac{1}{\tau_{21}}\right)N_2 - \left(\frac{1}{\tau_1}\right)N_1$$

$$\frac{dN_1}{dt} + \left(\frac{1}{\tau_1}\right)N_1 = +\left(\frac{1}{\tau_{21}}\right)N_2$$

$$\frac{dN_1}{dt} e^{t/\tau_1} + \left(\frac{1}{\tau_1}\right)N_1 e^{t/\tau_1} = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

$$\frac{d}{dt} (N_1 e^{t/\tau_1}) = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

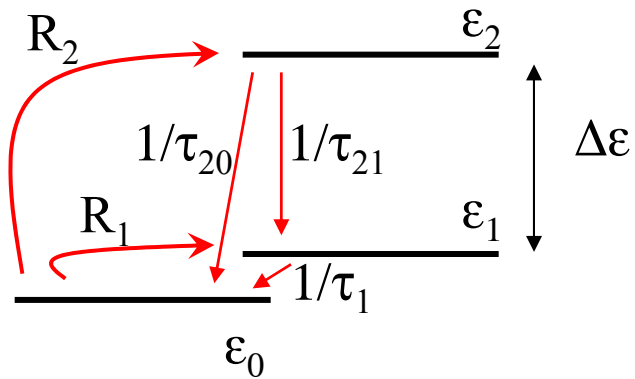
$$\int \frac{d}{dt} (N_1 e^{t/\tau_1}) = \int +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

$$N_1 e^{t/\tau_1} = \int +\left(\frac{1}{\tau_{21}}\right)R_{20}\tau_2(1 - e^{-t/\tau_2})e^{t/\tau_1}$$

Evaluating the integral on the r.h.s. gives...



Case One: I=0



Solution:

“Reservoir”

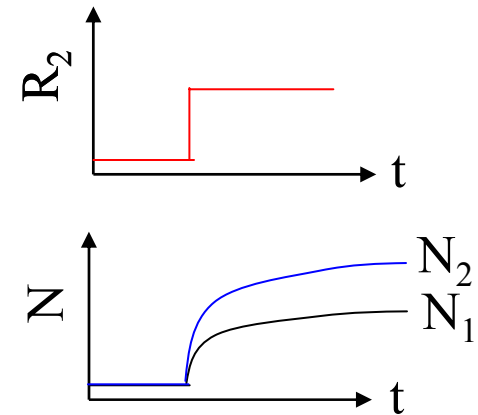
$$N_1(t) = \left(\frac{\tau_2}{\tau_{21}} \right) R_{20} \tau_1 \left\{ 1 + \frac{\tau_1 / \tau_2}{1 - \tau_1 / \tau_2} e^{-t/\tau_1} - \frac{1}{1 - \tau_1 / \tau_2} e^{-t/\tau_2} \right\}$$

$$N_2(t) = R_{20} \tau_2 \left(1 - e^{-t/\tau_2} \right)$$

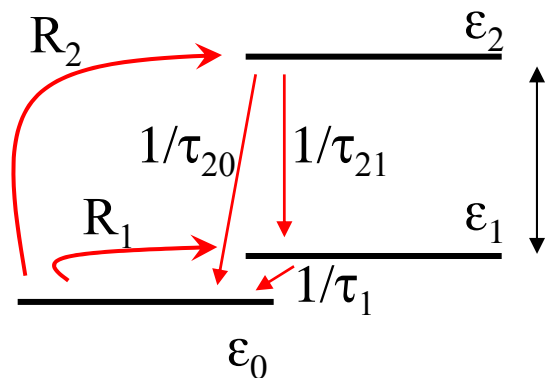
$$\tau_2 < \tau_{21}$$

needed for population inversion and gain!

Example: Nd:YAG 255 microseconds vs. 30 ns



Case Two: I is finite

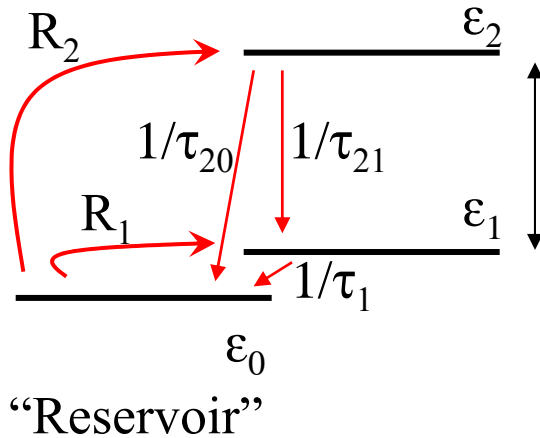


“Reservoir”

$$\Delta\epsilon \quad \frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)]$$



Case Two: I is finite

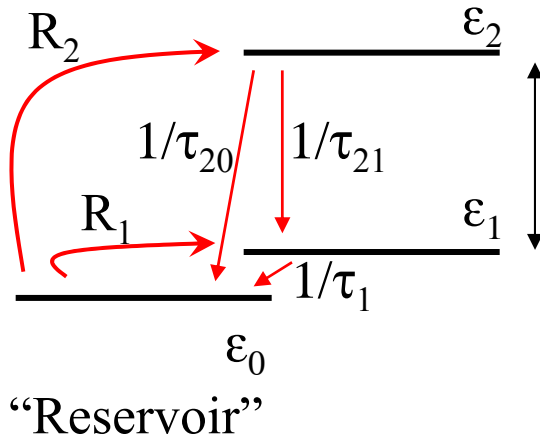


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because $\tau_1=0$



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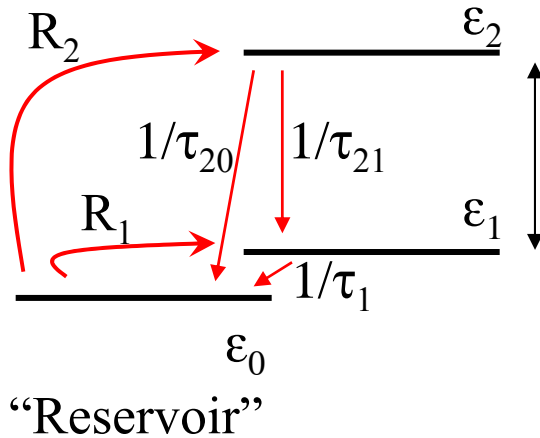
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$$\frac{dN_2}{dt} + \left(\frac{1}{\tau_2}\right)N_2 + \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2)] = R_2(t)$$



Case Two: I is finite



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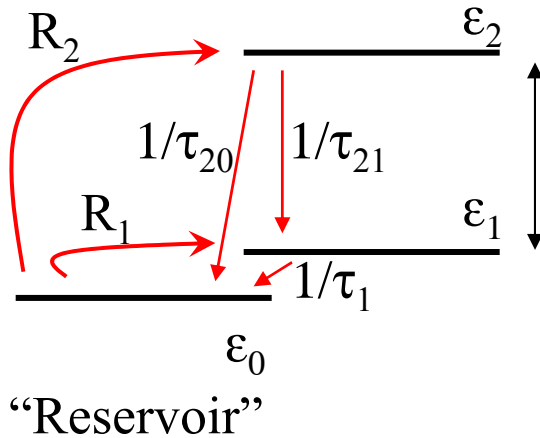
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$$I_s \equiv \frac{h\nu}{\sigma\tau_2} \quad \text{“Saturation intensity”}$$



Case Two: I is finite



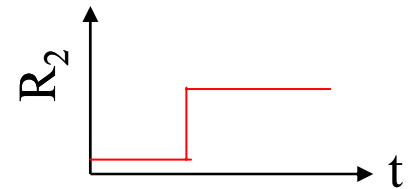
$$\Delta\epsilon \frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_v}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$

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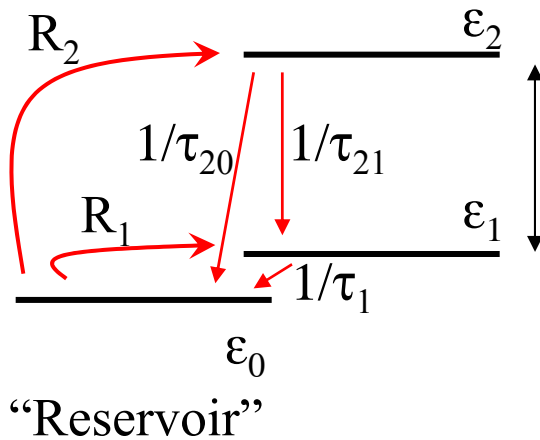
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$$\frac{dN_2}{dt} + \frac{1}{\tau_2} \left[1 + \frac{I_v}{I_s} \right] N_2 = R_2(t) = R_{20}u(t)$$



Case Two: I is finite



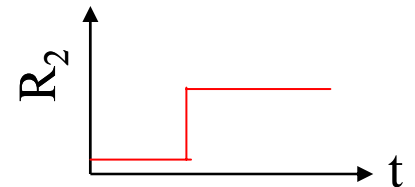
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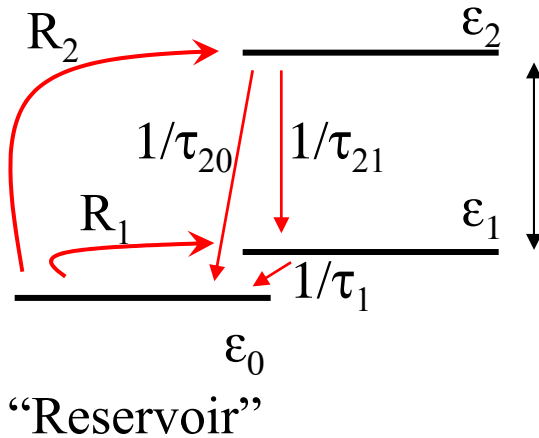


Solve as in case 1:

$$N_2(t) = \frac{R_{20}\tau_2}{1 + I_v/I_s} \left\{ 1 - e^{-\frac{t}{\tau_2}(1+I_v/I_s)} \right\}$$



Case Two: I is finite



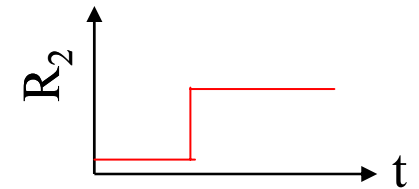
$$\Delta\varepsilon \frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_v}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)]$$

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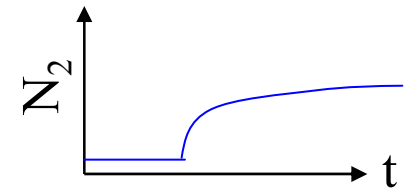
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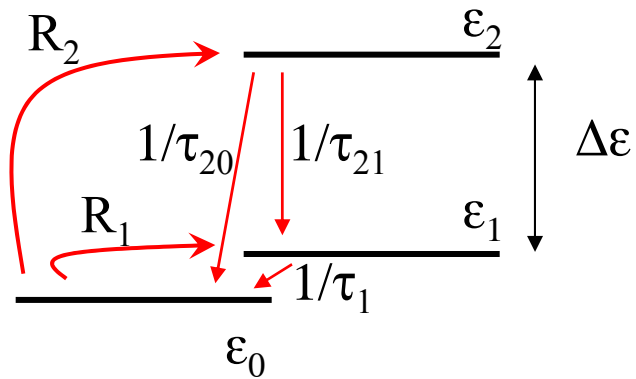


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Case One vs. Case Two



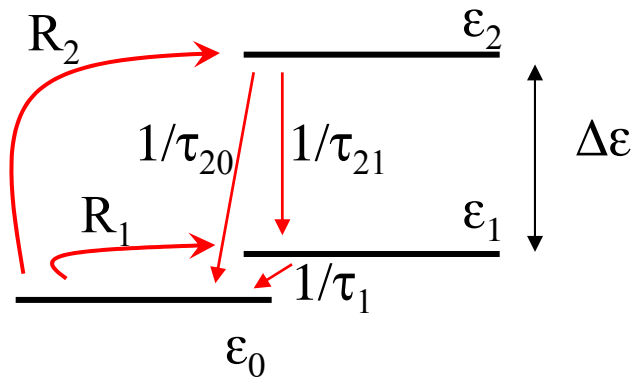
“Reservoir”

$$N_2(t) = R_{20} \tau_2 \left(1 - e^{-t/\tau_2} \right)$$

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Case One vs. Case Two



“Reservoir”

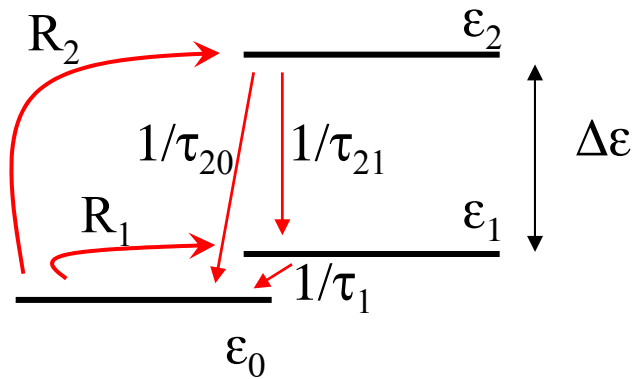
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1. For $I_v \ll I_s$, results are the same.



Case One vs. Case Two



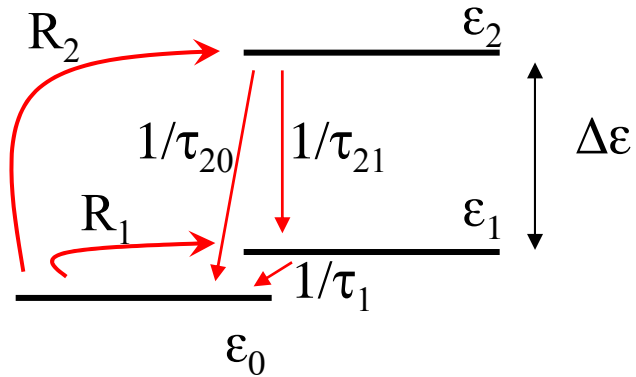
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1. For $I_v \ll I_s$, results are the same.
2. For $I_v = I_s$, results are different by exactly a factor of two.



Case One vs. Case Two



"Reservoir"

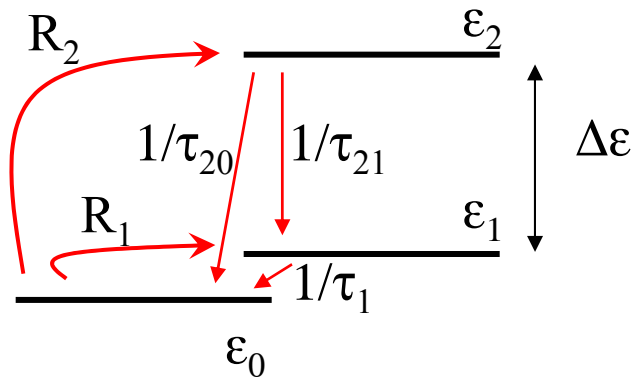
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1. For $I_v \ll I_s$, results are the same.
2. For $I_v = I_s$, results are different by exactly a factor of two.
3. For $I_v \gg I_s$, N_2 is much less than it would be if $I_v = 0$.



Case One vs. Case Two



“Reservoir”

$$N_2(t) = R_{20} \tau_2 \left(1 - e^{-t/\tau_2} \right)$$

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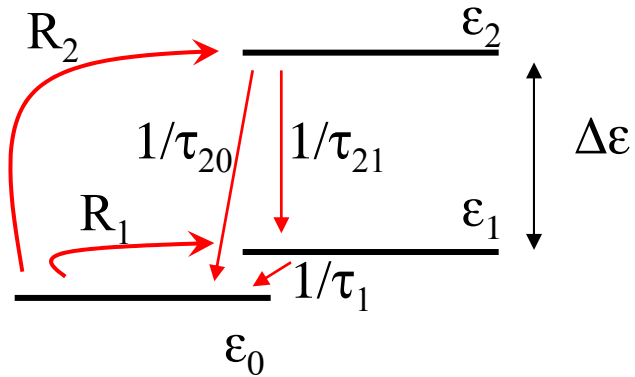
1. For $I_v \ll I_s$, results are the same.
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This gives rise *gain saturation* in lasers.



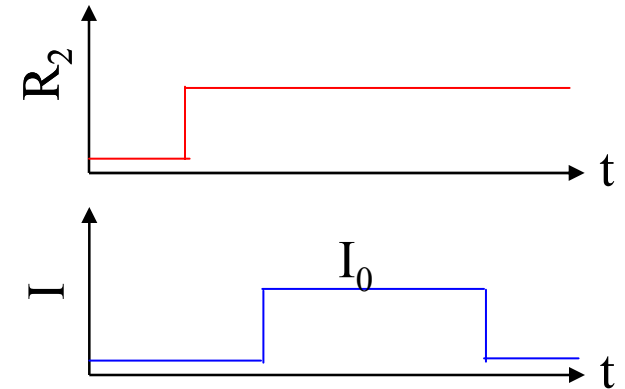
Case Three: I an impulse

Assume $\tau_1 = 0$ again. So $N_1 = 0$.



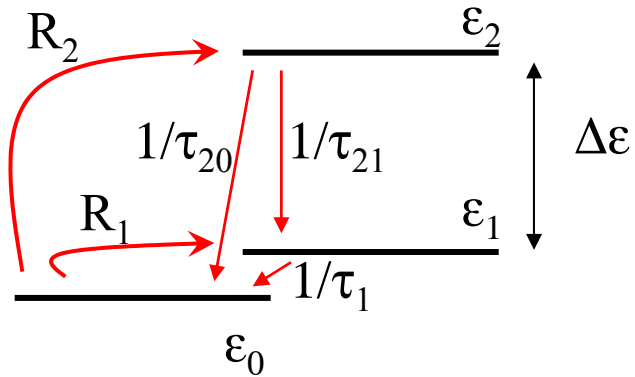
“Reservoir”

$$\frac{dN_2}{dt} + \frac{1}{\tau_2} \left[1 + \frac{I_v}{I_s} \right] N_2 = R_{20} u(t)$$



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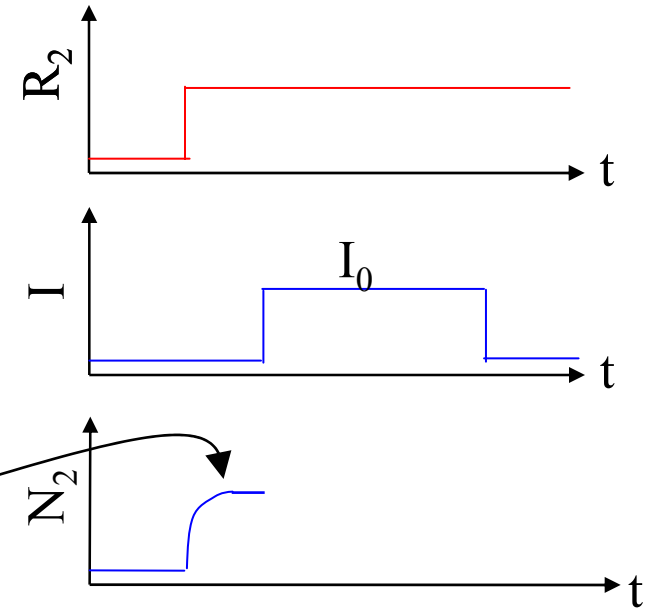


“Reservoir”

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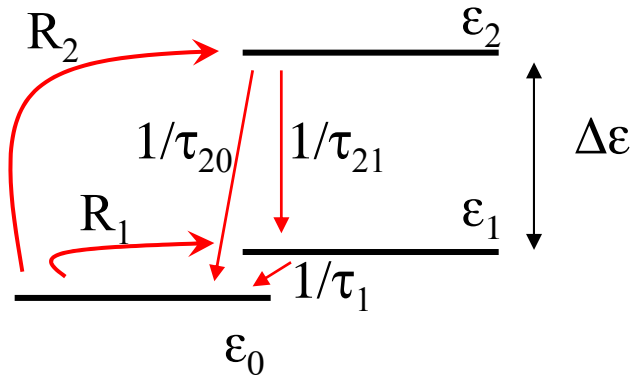
From case 1:

$$N_2(t) = R_{20} \tau_2 \left(1 - e^{-t/\tau_2} \right)$$



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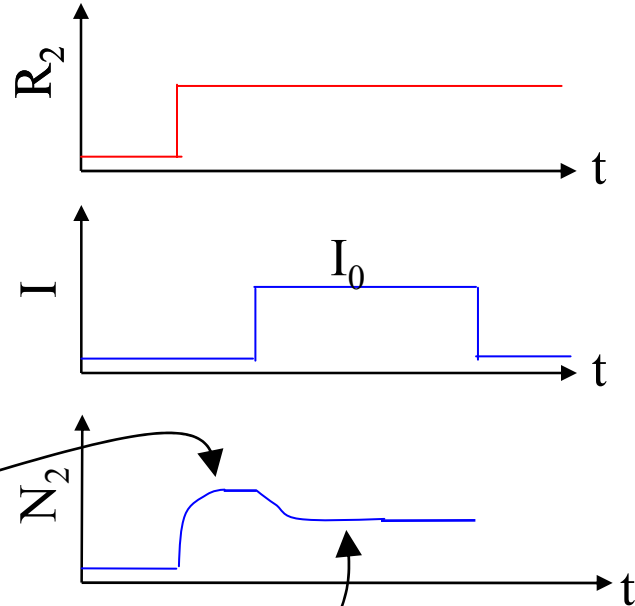


“Reservoir”

$$\frac{dN_2}{dt} + \frac{1}{\tau_2} \left[1 + \frac{I_v}{I_s} \right] N_2 = R_{20} u(t)$$

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Now we have to solve equation on top again, with new initial condition.

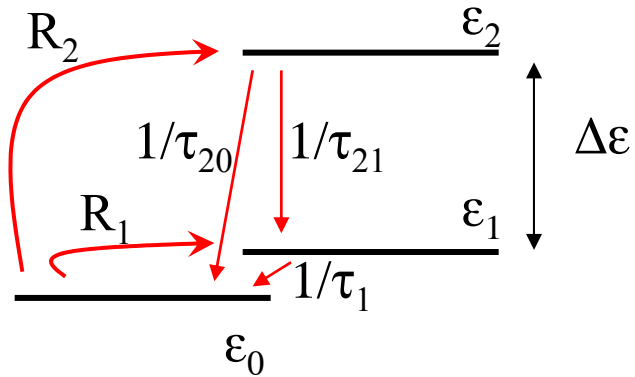
$$N_2(t) = \frac{R_{20} \tau_2}{1 + I_v / I_s} \left\{ \left(\frac{I_v}{I_s} \right) e^{-\frac{t}{\tau'_2}} + 1 \right\} \rightarrow \frac{R_{20} \tau_2}{1 + I_v / I_s} \quad \tau'_2 = \frac{\tau_2}{1 + I_0 / I_s}$$

So increase I decreases N_2 hence gain.



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Assume $\tau_1 = 0$ again. So $N_1 = 0$.

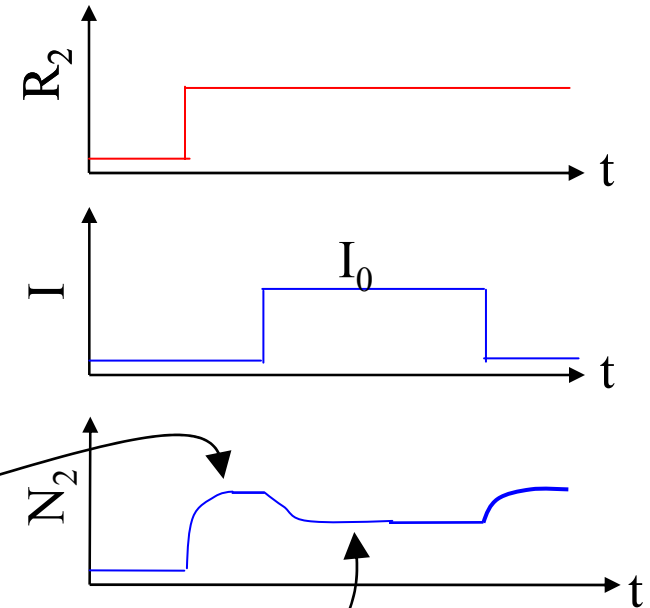


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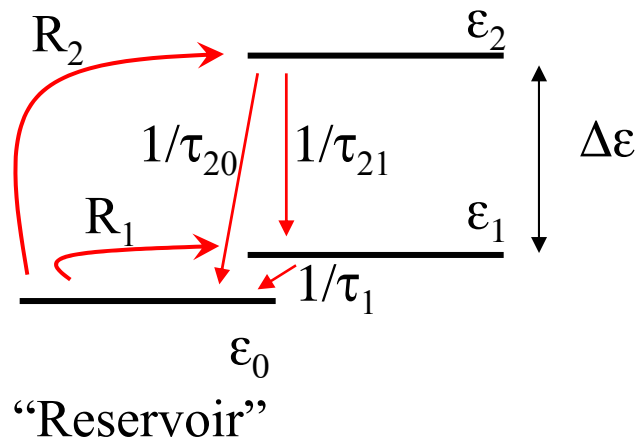
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Steady state: $dN/dt=0$



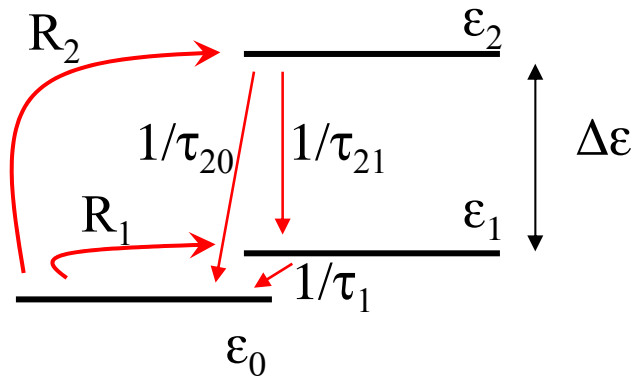
$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$

$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$



Steady state: $dN/dt=0$

Goal: Solve for N_1, N_2 :



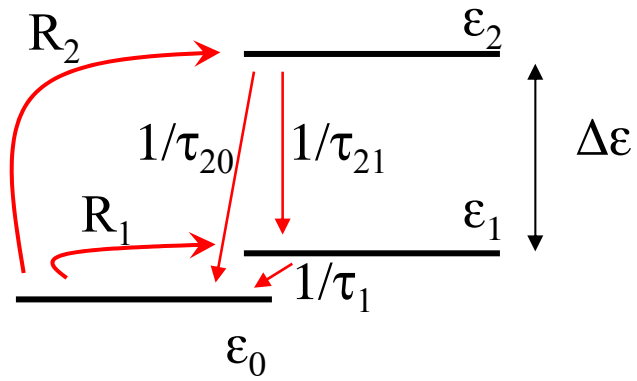
“Reservoir”



Steady state: $dN/dt=0$

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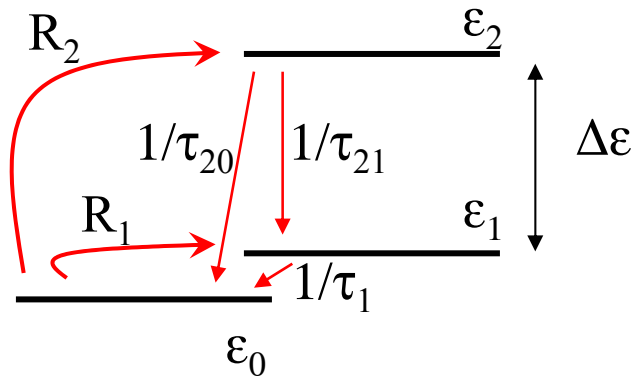
$$\frac{dN_2}{dt} = 0 \Rightarrow$$



“Reservoir”



Steady state: $dN/dt=0$



“Reservoir”

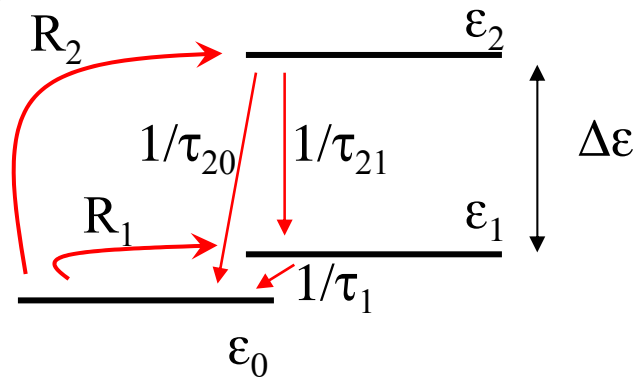
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Steady state: $dN/dt=0$



“Reservoir”

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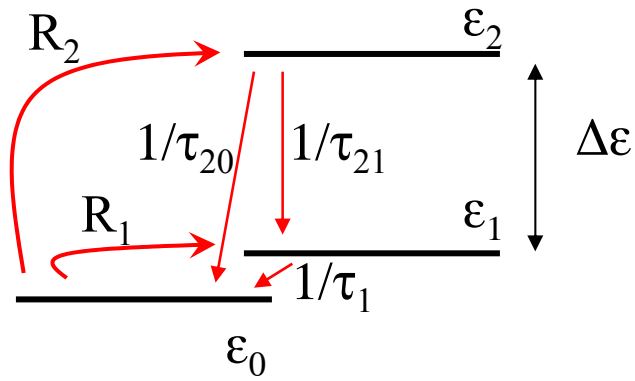
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$$R_2(t) = \left(\frac{1}{\tau_2} + \frac{I_\nu \sigma(\nu)}{h\nu}\right)N_2 - \frac{I_\nu \sigma(\nu)}{h\nu}N_1$$



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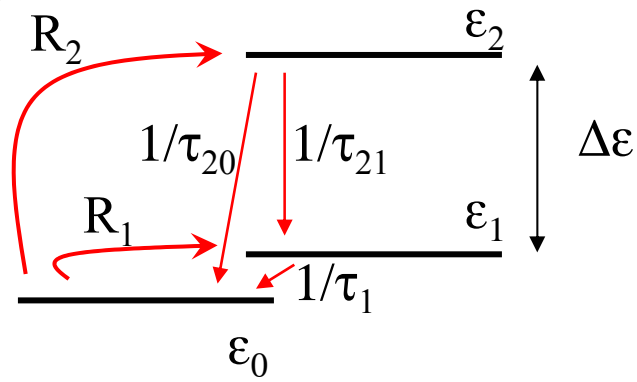
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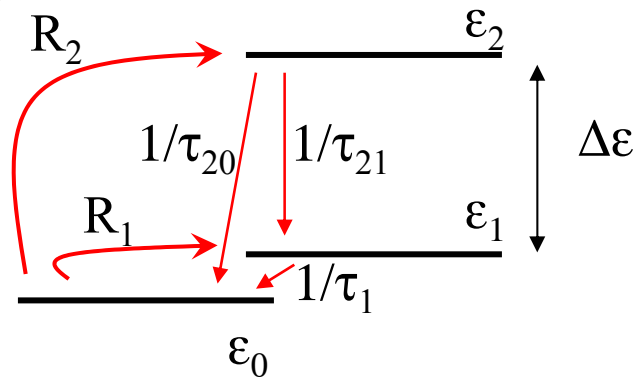
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Steady state: $dN/dt=0$



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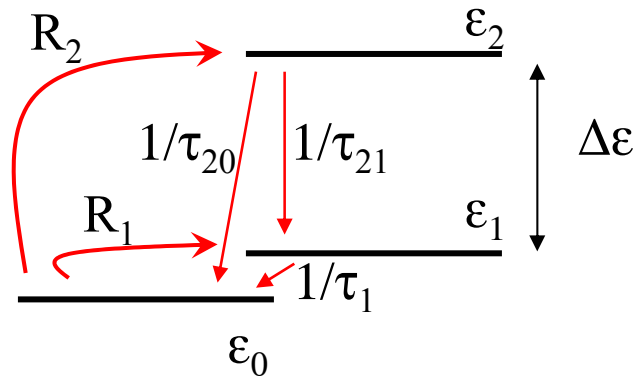
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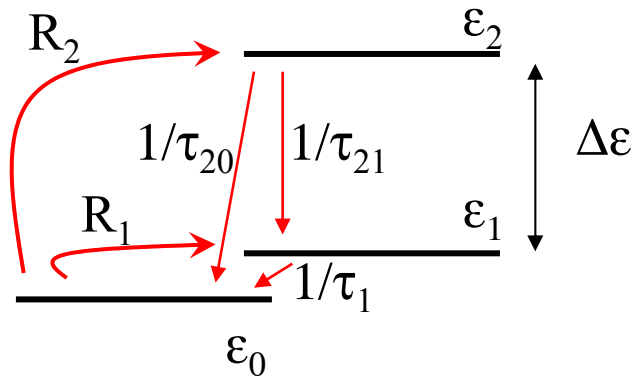
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Steady state: $dN/dt=0$



“Reservoir”

Goal: Solve for N_1, N_2 :

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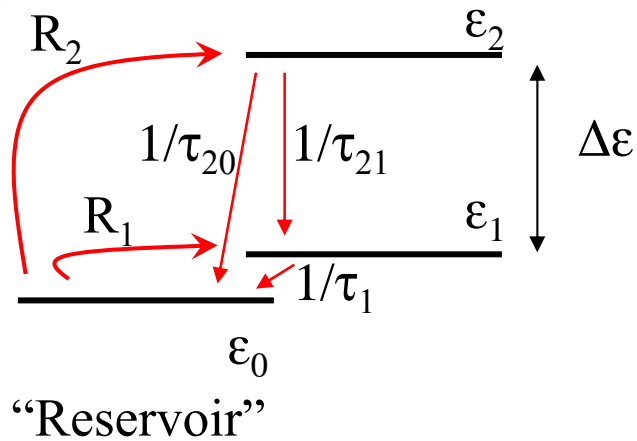
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Two equations, two unknowns.
Solve for N_1, N_2 .



Steady state: $dN/dt=0$

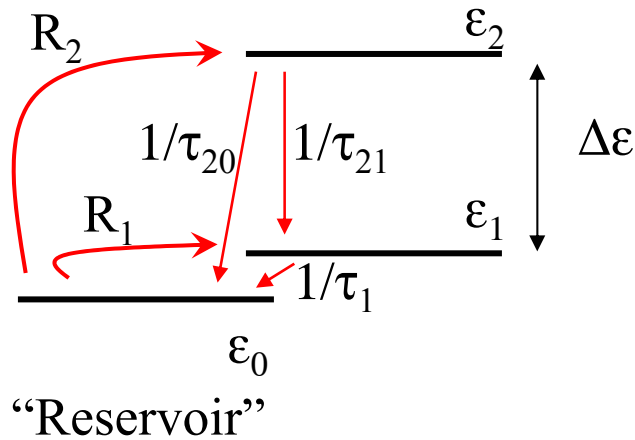


$$N_1 = \frac{R_2 \left(\frac{1}{\tau_1} + \frac{I_v \sigma(\nu)}{h\nu} \right) + R_1 \left(\frac{I_v \sigma(\nu)}{h\nu} \right)}{\frac{1}{\tau_1 \tau_2} \left[1 + \left(\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{\tau_{21}} \right) \frac{I_v \sigma(\nu)}{h\nu} \right]}$$

$$N_2 = \frac{R_1 \left(\frac{1}{\tau_2} + \frac{I_v \sigma(\nu)}{h\nu} \right) + R_2 \left(\frac{1}{\tau_{21}} + \frac{I_v \sigma(\nu)}{h\nu} \right)}{\frac{1}{\tau_1 \tau_2} \left[1 + \left(\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{\tau_{21}} \right) \frac{I_v \sigma(\nu)}{h\nu} \right]}$$



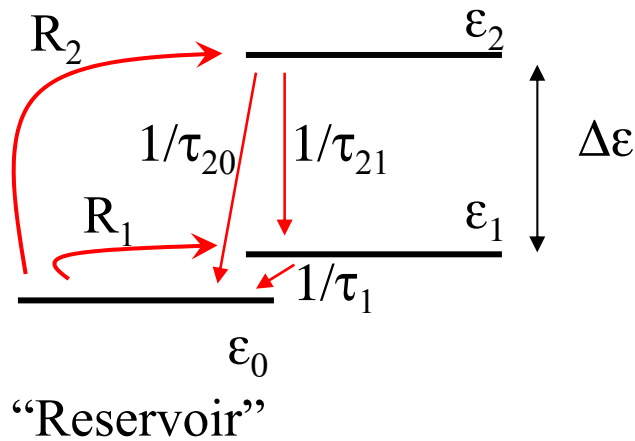
Steady state: $dN/dt=0$



$$N_2 - N_1 = \frac{R_2 \tau_2 \left(1 - \frac{\tau_1}{\tau_{21}} \right) - R_1 \tau_1}{1 + \left(\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{\tau_{21}} \right) \frac{I_\nu \sigma(\nu)}{h\nu}}$$



Steady state: $dN/dt=0$



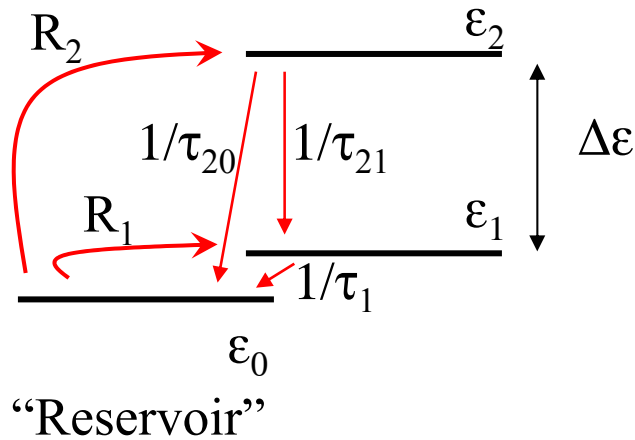
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$I=0$:

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Steady state: $dN/dt=0$



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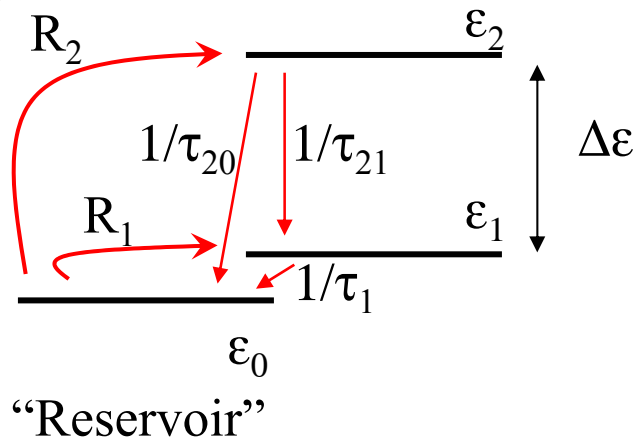
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$$\gamma_0 = \sigma(\nu) R_2 \tau_2 \left(1 - \frac{\tau_1}{\tau_{21}} \right) - R_1 \tau_1$$



Steady state: $dN/dt=0$



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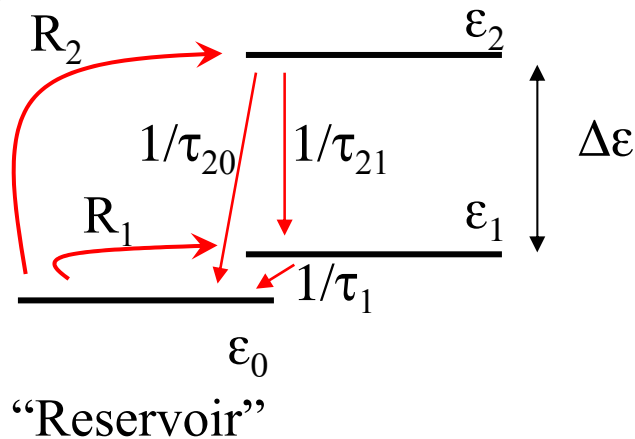
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$$\gamma_0(I) = \frac{\gamma_0}{1 + I_\nu / I_s}$$



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$$N_2 - N_1 = R_2 \tau_2 \left(1 - \frac{\tau_1}{\tau_{21}} \right) - R_1 \tau_1$$

$$\gamma = \sigma(\nu)(N_2 - N_1) \quad \gamma_0 = \sigma(\nu) R_2 \tau_2 \left(1 - \frac{\tau_1}{\tau_{21}} \right) - R_1 \tau_1$$

$$\gamma_0(I) = \frac{\gamma_0}{1 + I_\nu / I_s}$$



Saturation:

$$\frac{\partial I}{\partial z} = \gamma I_v \quad \gamma_0(I) = \frac{\gamma_0}{1 + I_v / I_s}$$



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Saturation:

$$\frac{\partial I}{\partial z} = \gamma I_v \quad \gamma_0(I) = \frac{\gamma_0}{1 + I_v / I_s}$$

$$G \equiv \frac{I_2}{I_1}$$

$$\frac{\partial I}{\partial z} = \frac{\gamma_0}{1 + I_v / I_s} I_v$$

$$\ln G + \frac{I_1}{I_s} (G - 1) = \gamma_0 l_g$$

$$\frac{1 + I_v / I_s}{I_v} \partial I = \gamma_0 \partial z$$

$$\int_{I_1}^{I_2} \frac{1 + I_v / I_s}{I_v} \partial I = \int_{z=0}^{z=l_g} \gamma_0 \partial z$$

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Case 1: $I_1 \ll I_s$

$$\ln G = \gamma_0 l_g \Rightarrow G = e^{\gamma_0 l_g}$$



Saturation:

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Case 1: $I_1 \ll I_s$

$$\ln G = \gamma_0 l_g \Rightarrow G = e^{\gamma_0 l_g}$$

Case 2: $I_1 \gg I_s$

$$I_2 = I_1 + \gamma_0 l_g I_s$$



Conclusions for today:

- 2-level system: No population inversion steady state
- Now we know that 3-level system can have gain.
- Now we know how saturation of gain comes about.
- Next week, we'll consider laser oscillation.

