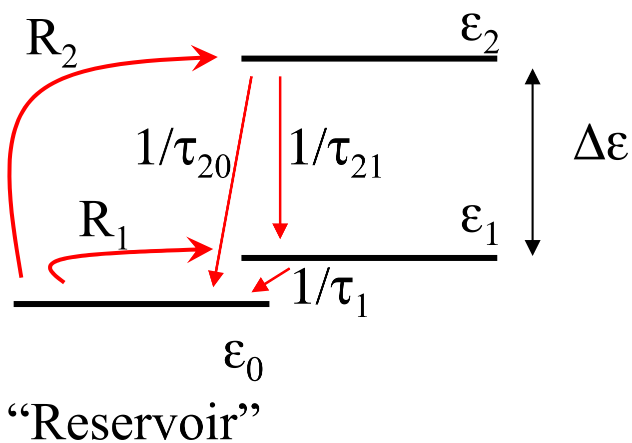
Lecture 6: Three-level rate equations

- Recall: No two-level laser (discuss)
- Three level laser: Let's try

Three-level rate equations:



R_1 and R_2 are the external "pump" rates.

$1/\tau_{20}$ is spontaneous emission rate 2->0

$1/\tau_{21}$ is spontaneous emission rate 2->1

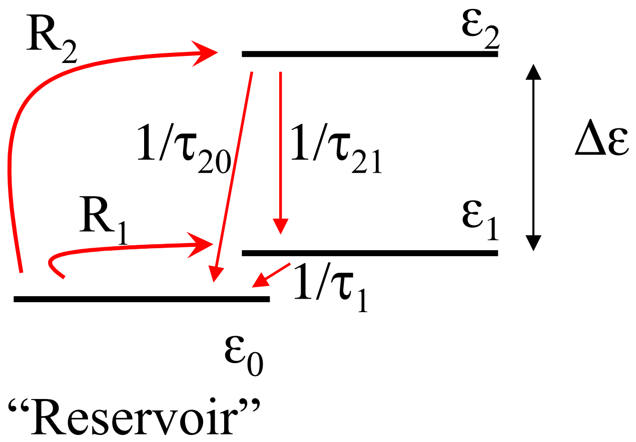
$1/\tau_2 = 1/\tau_{21} + 1/\tau_{20}$ spont em rate 2->(anything)

$1/\tau_1$ is spontaneous emission rate 1->0

Goal: Use R_2 to achieve steady-state population inversion.

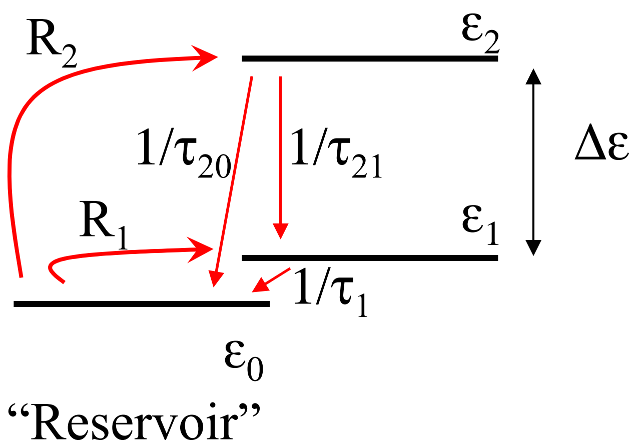
Guess:
Want lifetime in state 2 very long,
Want lifetime in state 1 very short.

Three-level rate equations:



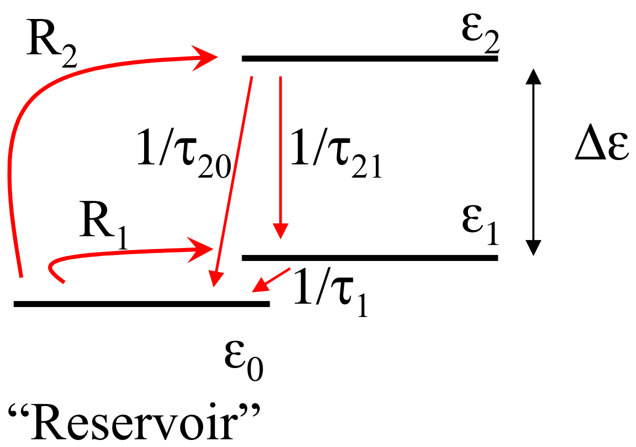
$$\frac{dN_2}{dt} = R_2(t)$$

Three-level rate equations:



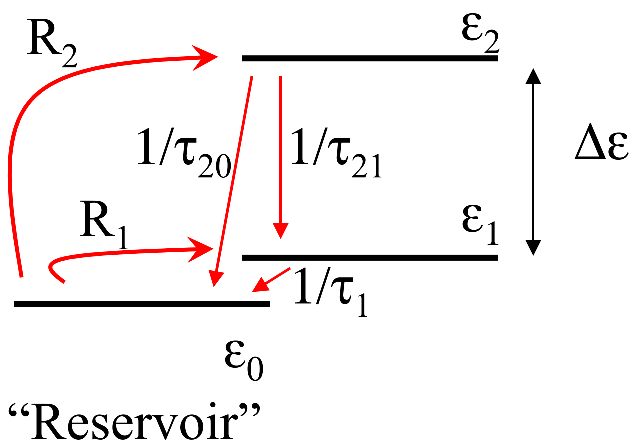
$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right) N_2$$

Three-level rate equations:



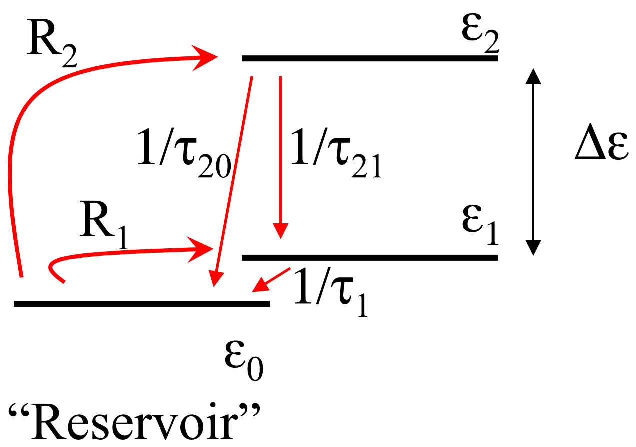
$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - N_1)\right]$$

Three-level rate equations:



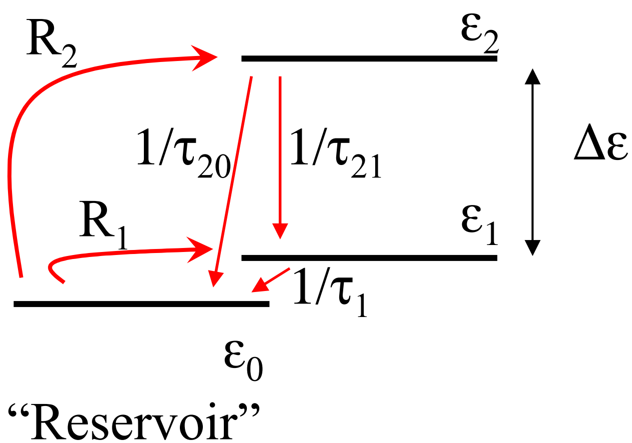
$$\boxed{\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right) N_2 - \frac{I_\nu}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - N_1)\right]}$$

Three-level rate equations:



$$\boxed{\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - N_1)\right] - \left(\frac{1}{\tau_1}\right)N_1}$$

Three-level rate equations:



$$\boxed{\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - N_1)\right]}$$

$$\boxed{\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - N_1)\right] - \left(\frac{1}{\tau_1}\right)N_1}$$

Four cases to consider:

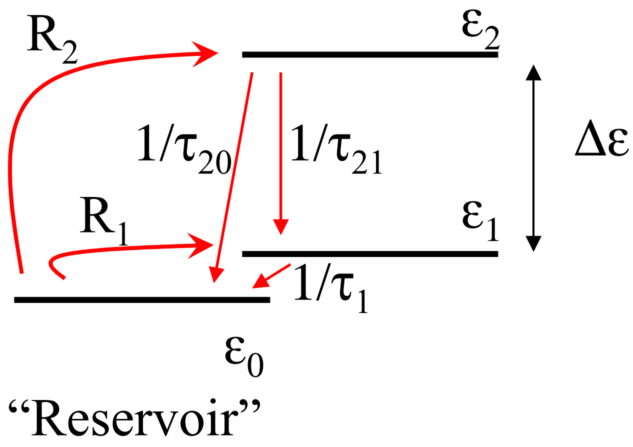
1) $I = 0$

2) I finite, $\tau_1 = 0$

3) I is pulsed.

4) General conditions for steady state.

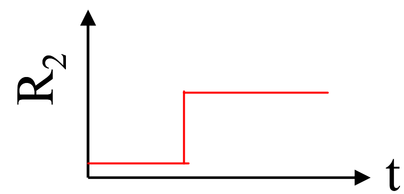
Case One: I=0

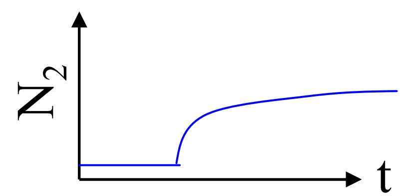


$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \underbrace{\frac{I_\nu}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - N_1)\right]}_{0}$$

$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2$$

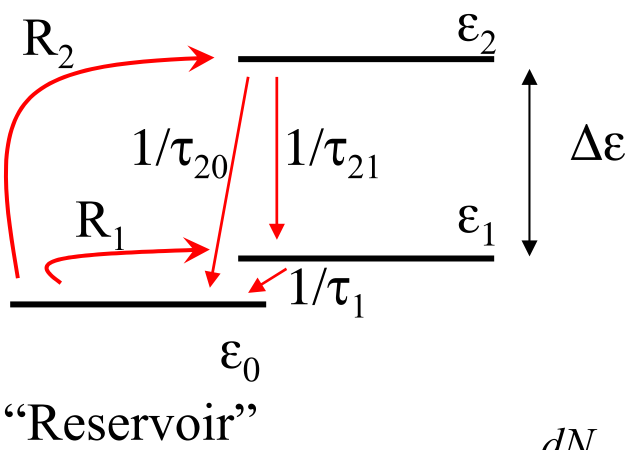
Let: $R_2(t) = R_{20}u(t)$ $\qquad N_2(0) = 0$





$$N_2(t) = \int \frac{dN_2}{dt}dt = R_{20}\tau_2\left(1 - e^{-t/\tau_2}\right)$$

Case One: I=0



$$\frac{dN_1}{dt} = \underbrace{R_1(t)}_{0} + \left(\frac{1}{\tau_{21}}\right)N_2 + \underbrace{\frac{I_\nu}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - N_1)\right]}_{0} - \left(\frac{1}{\tau_1}\right)N_1$$

$$\frac{dN_1}{dt} = +\left(\frac{1}{\tau_{21}}\right)N_2 - \left(\frac{1}{\tau_1}\right)N_1$$

$$\frac{dN_1}{dt} + \left(\frac{1}{\tau_1}\right)N_1 = +\left(\frac{1}{\tau_{21}}\right)N_2$$

$$\frac{dN_1}{dt}e^{t/\tau_1} + \left(\frac{1}{\tau_1}\right)N_1 e^{t/\tau_1} = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

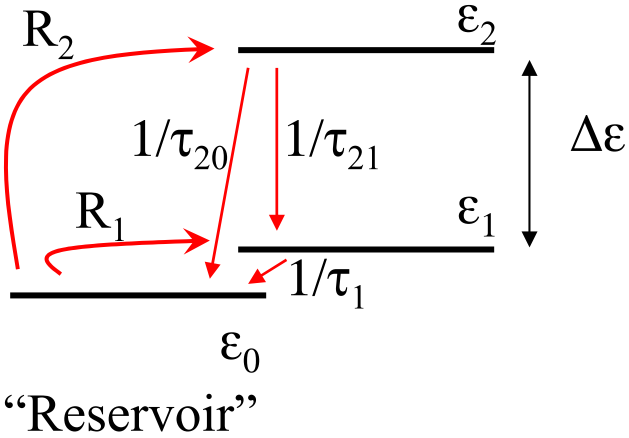
$$\frac{d}{dt}\left(N_1 e^{t/\tau_1}\right) = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

$$\int \frac{d}{dt}\left(N_1 e^{t/\tau_1}\right) = \int +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

$$N_1 e^{t/\tau_1} = \int +\left(\frac{1}{\tau_{21}}\right)R_{20}\tau_2\left(1 - e^{-t/\tau_2}\right)e^{t/\tau_1}$$

Evaluating the integral on the r.h.s. gives…

Case One: I=0



Solution:

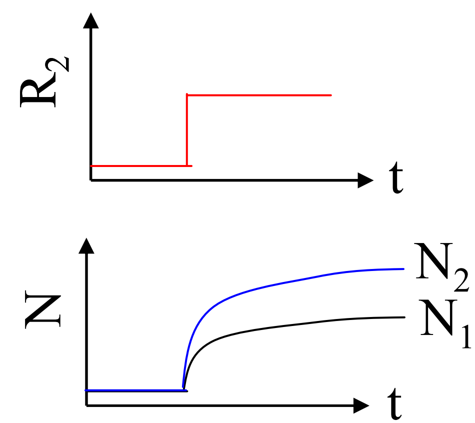
$$N_1(t) = \left(\frac{\tau_2}{\tau_{21}}\right) R_{20}\tau_1 \left\{ 1 + \frac{\tau_1/\tau_2}{1-\tau_1/\tau_2} e^{-t/\tau_1} - \frac{1}{1-\tau_1/\tau_2} e^{-t/\tau_2} \right\}$$

$$N_2(t) = R_{20}\tau_2\left(1 - e^{-t/\tau_2}\right)$$

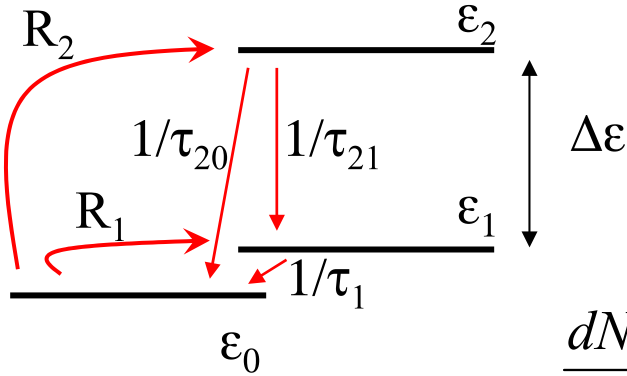
$$\tau_2 > \tau_1$$

needed for population inversion and gain!

Example: Nd:YAG 255 microseconds vs. 30 ns



Case Two: I is finite



“Reservoir”

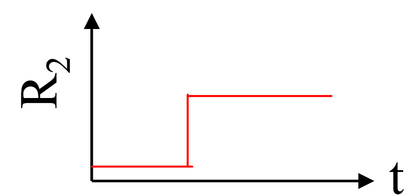
$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - \cancelto{0}{N_1})\right]$$

because $\tau_1=0$

$$\frac{dN_2}{dt} + \left(\frac{1}{\tau_2}\right)N_2 + \frac{I_\nu}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2)\right] = R_2(t)$$

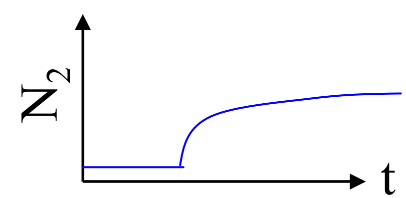
$$I_s \equiv \frac{h\upsilon}{\sigma\tau_2}$$ “Saturation intensity”

$$\frac{dN_2}{dt} + \frac{1}{\tau_2}\left[1 + \frac{I_\nu}{I_s}\right]N_2 = R_2(t) = R_{20}u(t)$$

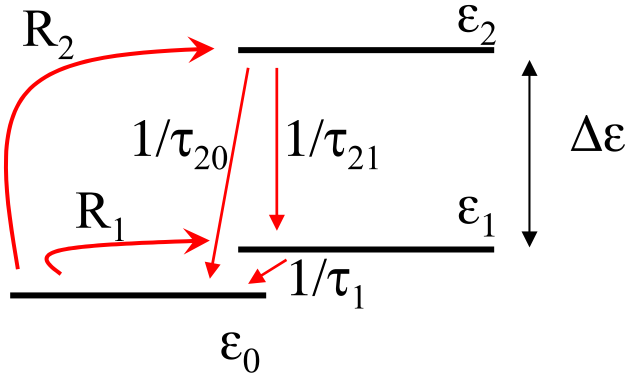


Solve as in case 1:

$$\boxed{N_2(t) = \frac{R_{20}\tau_2}{1 + I_\nu / I_s}\left\{1 - e^{-\frac{t}{\tau_2}(1 + I_\nu / I_s)}\right\}}$$



Case One vs. Case Two



“Reservoir”

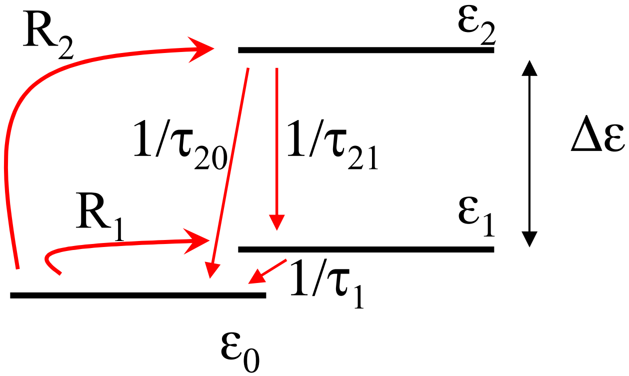
$$N_2(t) = R_{20}\tau_2\left(1 - e^{-t/\tau_2}\right)$$

$$N_2(t) = \frac{R_{20}\tau_2}{1 + I_\nu / I_s}\left\{1 - e^{-\frac{t}{\tau_2}(1 + I_\nu / I_s)}\right\}$$

1. For $I_\nu << I_s$, results are the same.
2. For $I_\nu = I_s$, results are different by exactly a factor of two.
3. For $I_\nu >> I_s$, N_2 is much less than it would be if $I_\nu = 0$.

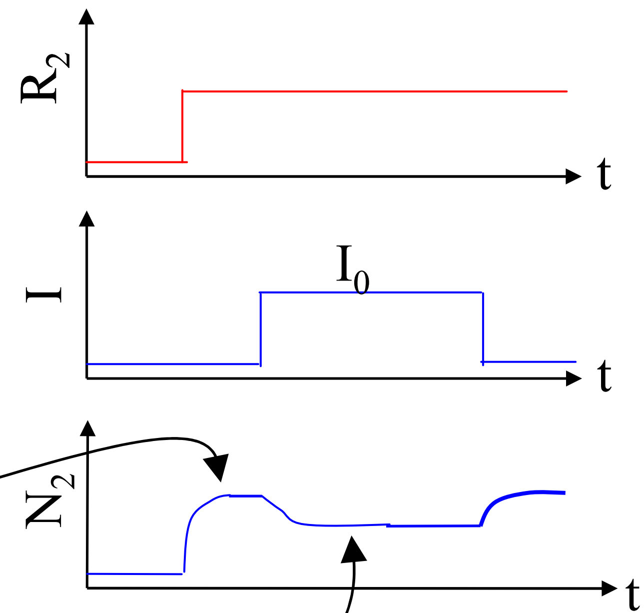
This gives rise *gain saturation* in lasers.

Case Three: I an impulse



"Reservoir"

Assume $\tau_1 = 0$ again. So $N_1 = 0$.



$$\frac{dN_2}{dt} + \frac{1}{\tau_2}\left[1 + \frac{I_v}{I_s}\right] N_2 = R_{20} u(t)$$

From case 1:

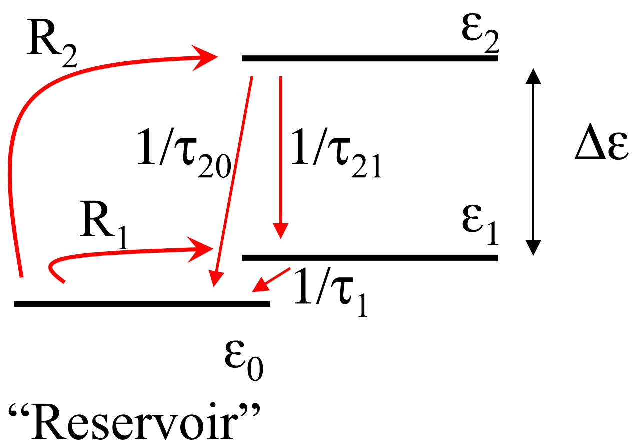
$$N_2(t) = R_{20}\tau_2\left(1 - e^{-t/\tau_2}\right)$$

Now we have to solve equation on top again, with new initial condition.

$$N_2(t) = \frac{R_{20}\tau_2}{1 + I_v/I_s}\left\{\left(\frac{I_v}{I_s}\right) e^{-\frac{t}{\tau'_2}} + 1\right\} \rightarrow \frac{R_{20}\tau_2}{1 + I_v/I_s} \qquad \tau'_2 = \frac{\tau_2}{1 + I_0/I_s}$$

So increase I decreases N_2 hence gain.

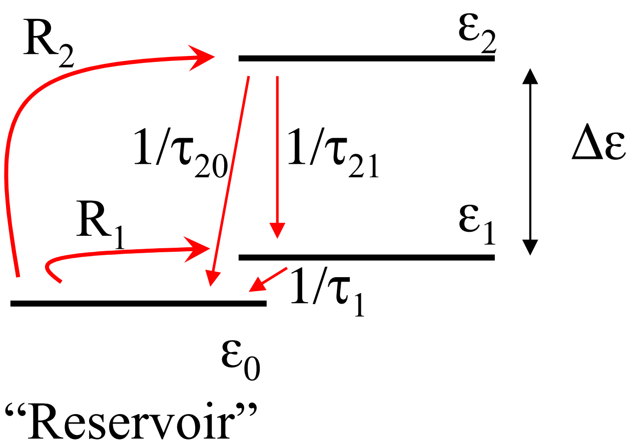
Steady state: dN/dt=0



$$\boxed{\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right) N_2 - \frac{I_\nu}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - N_1)\right]}$$

$$\boxed{\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right) N_2 + \frac{I_\nu}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - N_1)\right] - \left(\frac{1}{\tau_1}\right) N_1}$$

Steady state: dN/dt=0



Goal: Solve for N_1, N_2:

$$\frac{dN_2}{dt} = 0 \Rightarrow$$

$$0 = R_2(t) - \left(\frac{1}{\tau_2}\right) N_2 - \frac{I_v}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - N_1)\right]$$

$$\boxed{R_2(t) = \left(\frac{1}{\tau_2} + \frac{I_v\sigma(\upsilon)}{h\upsilon}\right) \boxed{N_2} - \frac{I_v\sigma(\upsilon)}{h\upsilon} \boxed{N_1}}$$

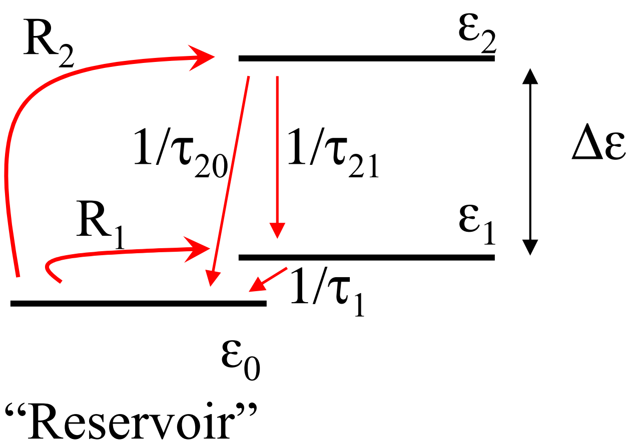
$$\frac{dN_1}{dt} = 0 \Rightarrow$$

$$0 = R_1(t) + \left(\frac{1}{\tau_{21}}\right) N_2 + \frac{I_v}{h\upsilon}\left[\sigma(\upsilon)\cdot(N_2 - N_1)\right] - \left(\frac{1}{\tau_1}\right) N_1$$

Two equations, two unknowns.
Solve for N_1, N_2.

$$\boxed{R_1(t) = -\left(\frac{1}{\tau_{21}} + \frac{I_v\sigma(\upsilon)}{h\upsilon}\right) \boxed{N_2} + \left(\frac{I_v\sigma(\upsilon)}{h\upsilon} + \frac{1}{\tau_1}\right) \boxed{N_1}}$$

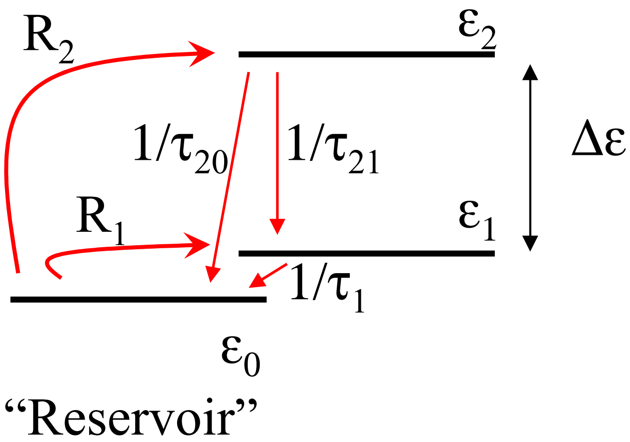
Steady state: dN/dt=0



$$N_1 = \frac{R_2\left(\frac{1}{\tau_1} + \frac{I_v \sigma(\upsilon)}{h\upsilon}\right) + R_1\left(\frac{I_v \sigma(\upsilon)}{h\upsilon}\right)}{\frac{1}{\tau_1\tau_2}\left[1 + \left(\tau_1 + \tau_2 - \frac{\tau_1\tau_2}{\tau_{21}}\right)\frac{I_v \sigma(\upsilon)}{h\upsilon}\right]}$$

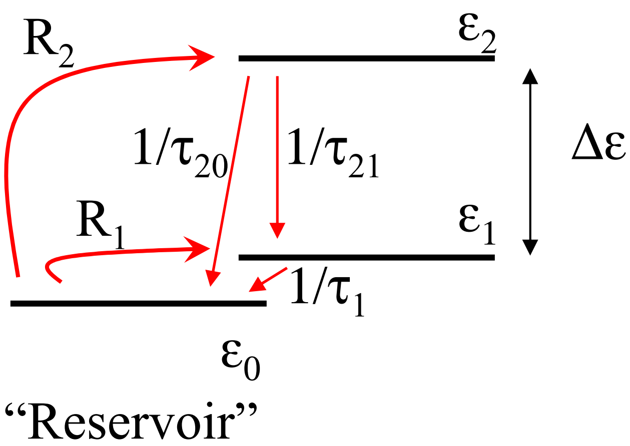
$$N_2 = \frac{R_1\left(\frac{1}{\tau_2} + \frac{I_v \sigma(\upsilon)}{h\upsilon}\right) + R_2\left(\frac{1}{\tau_{21}} + \frac{I_v \sigma(\upsilon)}{h\upsilon}\right)}{\frac{1}{\tau_1\tau_2}\left[1 + \left(\tau_1 + \tau_2 - \frac{\tau_1\tau_2}{\tau_{21}}\right)\frac{I_v \sigma(\upsilon)}{h\upsilon}\right]}$$

Steady state: dN/dt=0



$$N_2 - N_1 = \frac{R_2\tau_2\left(1-\frac{\tau_1}{\tau_{21}}\right) - R_1\tau_1}{1+\left(\tau_1+\tau_2-\frac{\tau_1\tau_2}{\tau_{21}}\right)\frac{I_\nu\sigma(\upsilon)}{h\upsilon}}$$

Steady state: dN/dt=0

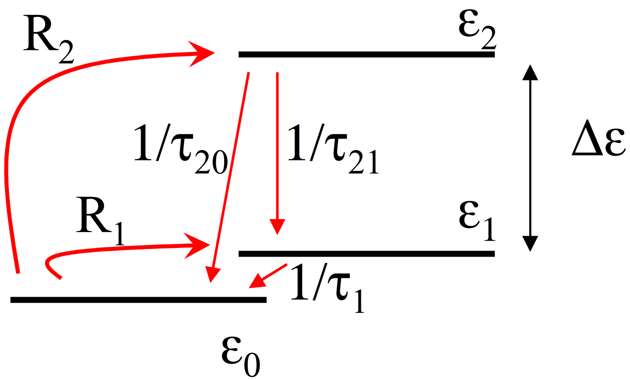


$$\boxed{N_2 - N_1 = \frac{R_2\tau_2\left(1-\dfrac{\tau_1}{\tau_{21}}\right) - R_1\tau_1}{1+\left(\tau_1+\tau_2-\dfrac{\tau_1\tau_2}{\tau_{21}}\right)\dfrac{I_\nu\sigma(\upsilon)}{h\upsilon}}}$$

I=0:

$$N_2 - N_1 = R_2\tau_2\left(1-\frac{\tau_1}{\tau_{21}}\right) - R_1\tau_1$$

Steady state: dN/dt=0



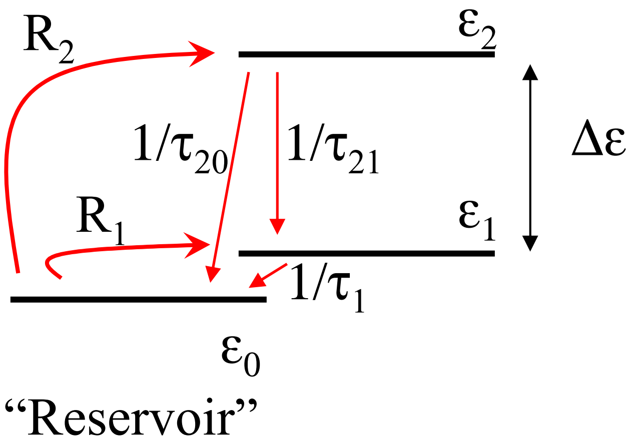
$$N_2 - N_1 = \frac{R_2\tau_2\left(1-\frac{\tau_1}{\tau_{21}}\right) - R_1\tau_1}{1+\left(\tau_1+\tau_2-\frac{\tau_1\tau_2}{\tau_{21}}\right)\frac{I_v\sigma(\upsilon)}{h\upsilon}}$$

I=0:

$$N_2 - N_1 = R_2\tau_2\left(1-\frac{\tau_1}{\tau_{21}}\right) - R_1\tau_1$$

$$\gamma_0 = \sigma(\upsilon)R_2\tau_2\left(1-\frac{\tau_1}{\tau_{21}}\right) - R_1\tau_1$$

Steady state: dN/dt=0



$$\boxed{N_2 - N_1 = \frac{R_2\tau_2\left(1-\dfrac{\tau_1}{\tau_{21}}\right) - R_1\tau_1}{1+\left(\tau_1+\tau_2-\dfrac{\tau_1\tau_2}{\tau_{21}}\right)\dfrac{I_\nu\sigma(\upsilon)}{h\upsilon}}}$$

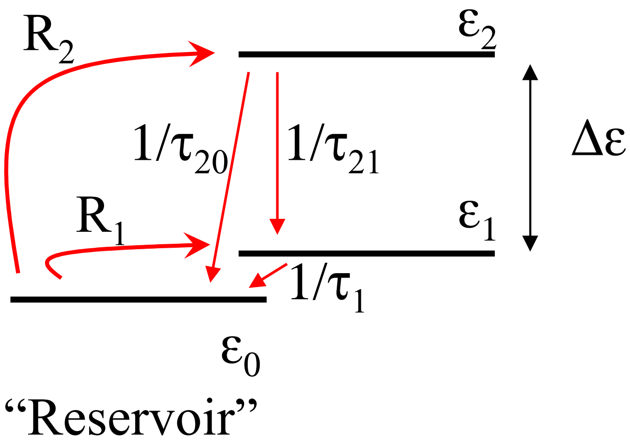
I=0:

$$N_2 - N_1 = R_2\tau_2\left(1-\frac{\tau_1}{\tau_{21}}\right) - R_1\tau_1$$

$$\gamma_0 = \sigma(\upsilon)R_2\tau_2\left(1-\frac{\tau_1}{\tau_{21}}\right) - R_1\tau_1$$

$$\gamma_0(I) = \frac{\gamma_0}{1+I_\upsilon / I_s}$$

Steady state: dN/dt=0



$$\boxed{N_2 - N_1 = \frac{R_2\tau_2\left(1-\frac{\tau_1}{\tau_{21}}\right)-R_1\tau_1}{1+\left(\tau_1+\tau_2-\frac{\tau_1\tau_2}{\tau_{21}}\right)\frac{I_\nu\sigma(\upsilon)}{h\upsilon}}}$$

$$\frac{\partial I}{\partial z} = \gamma I_\nu$$

I=0:

$$N_2 - N_1 = R_2\tau_2\left(1-\frac{\tau_1}{\tau_{21}}\right)-R_1\tau_1$$

$$\gamma = \sigma(\upsilon)(N_2 - N_1)$$

$$\gamma_0 = \sigma(\upsilon)R_2\tau_2\left(1-\frac{\tau_1}{\tau_{21}}\right)-R_1\tau_1$$

$$\gamma_0(I) = \frac{\gamma_0}{1+I_\upsilon/I_s}$$

Saturation:

$$\frac{\partial I}{\partial z} = \gamma I_\nu \qquad \gamma_0(I) = \frac{\gamma_0}{1 + I_\nu / I_s}$$

$$\frac{\partial I}{\partial z} = \frac{\gamma_0}{1 + I_\nu / I_s} I_\nu$$

$$\frac{1 + I_\nu / I_s}{I_\nu} \partial I = \gamma_0 \partial z$$

$$\int_{I_1}^{I_2} \frac{1 + I_\nu / I_s}{I_\nu} \partial I = \int_{z=0}^{z=l_g} \gamma_0 \partial z$$

$$\ln \frac{I_2}{I_1} + \frac{I_2 - I_1}{I_s} = \gamma_0 l_g$$

$$G \equiv \frac{I_2}{I_1}$$

$$\ln G + \frac{I_1}{I_s}(G - 1) = \gamma_0 l_g$$

Case 1: $I_1 << I_s$

$$\ln G = \gamma_0 l_g \Rightarrow G = e^{\gamma_0 l_g}$$

Case 2: $I_1 >> I_s$

$$I_2 = I_1 + \gamma_0 l_g I_s$$

Conclusions for today:

- 2-level system: No population inversion steady state
- Now we know that 3-level system can have gain.
- Now we know how saturation of gain comes about.
- Next week, we'll consider laser oscillation.