Lecture 3: Fields in a box

- Review of last week
- Electromagnetic modes in a box (classical)
- Single vs. mutlimode lasing
- We won't really get to photons yet.
- Reading: Ch. 7 of Verdeyen

Maxwell's equations:

$$\vec{\nabla} \cdot \vec{e} = \rho / \varepsilon_0 \qquad \vec{\nabla} \cdot \vec{b} = 0$$

$$\vec{\nabla} x \vec{e} = -\frac{\partial \vec{b}}{\partial t} \qquad \vec{\nabla} x \vec{b} = \mu_0 \varepsilon_0 \frac{\partial \vec{e}}{\partial t} + \mu_0 \vec{j}$$

Wave equation (ρ,j=0):

$$\nabla^2 \vec{e} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{e} \qquad \nabla^2 \vec{b} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{b}$$

Plane waves (ρ,j=0):

$$\vec{e}(\vec{r}, t) = \vec{e}_0 \, e^{i(\vec{k} \cdot \vec{r} \pm \omega t)}$$

Note ±:
+ is prop in +k dir.
- is prop in –k dir.

This week:

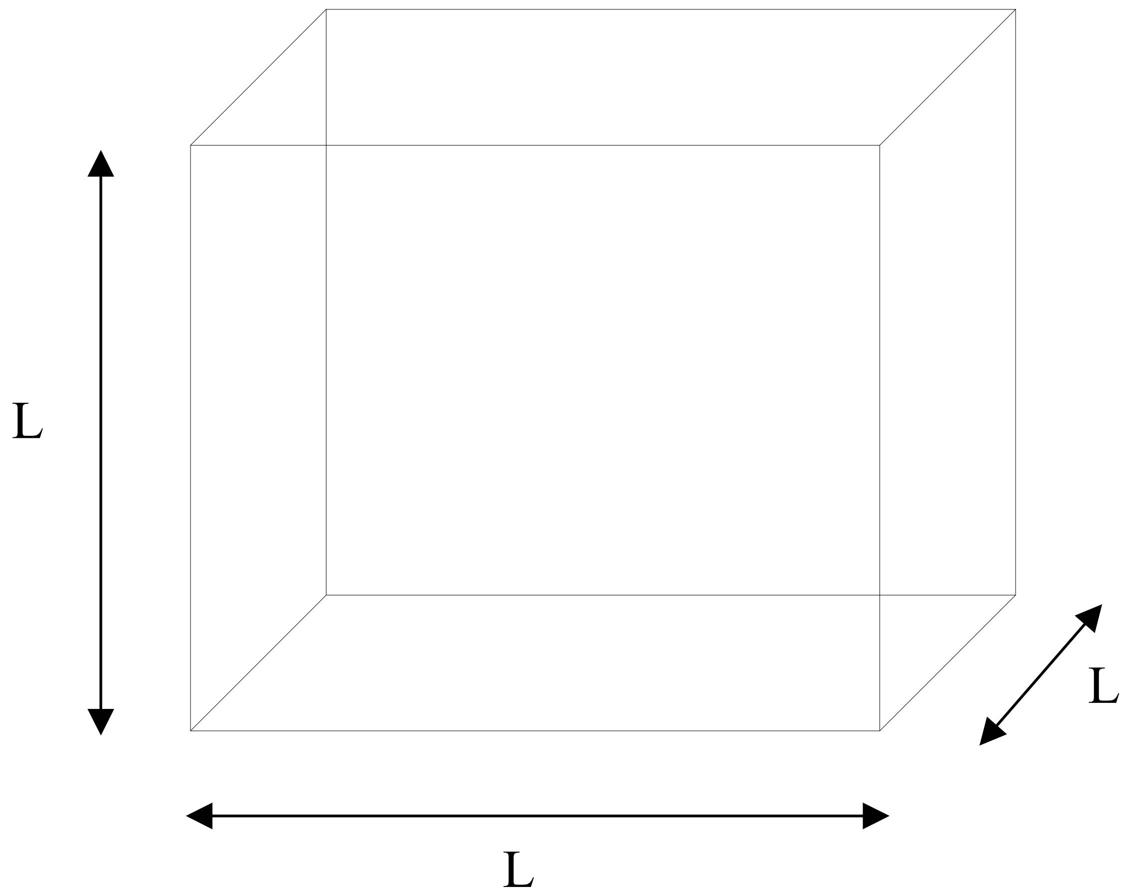
- Last week, fields were infinite in extent
- We also considered reflection off of boundaries that were infinite in extent
- Alluded to beams, and confining them
- Today we will discuss confinement further
- Motivation:
  1. Understand confined E-M fields in simplest confinement (box). *Pay attention: A laser is like a 1 dimensional box!!!*
  2. Understand blackbody radiation and photons

“A comment…”

- As a creative human being, you should strive to “think outside of the box.”

- But, for today’s lecture …

Confined fields: A metal box

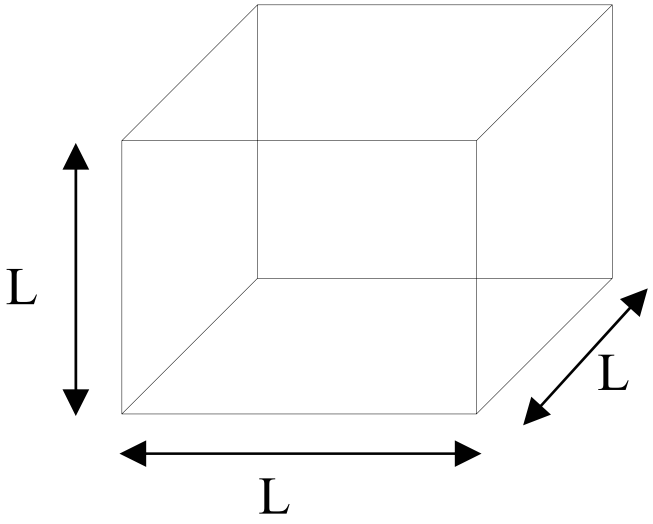


Goal: Find allowed E(r,t) and B(r,t) (i.e. obeys M.E. and B.C.)

Note: Complete opposite of plane waves

Note: Beams and fibers: confinement in only 2 dimensions

Confined fields: A metal box



Program:

1. Calculate E(r,t), B(r,t) inside box
2. Calculate energy stored inside box
3. Relate energy to temperature
4. Determine E(r,t), B(r,t) as a function temperature.
5. Let L go to infinity.

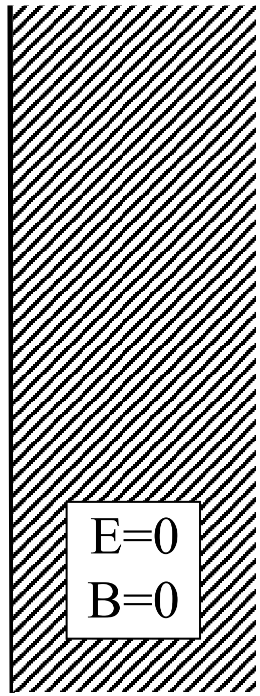
Along the way we will discuss *standing waves*.

We will need to quantize E,H
in order to get physically meaningful results: concept of a *photon*

Boundary conditions at a metal:

Inside a perfect metal, E=0 and B=0 by definition.

Vacuum Metal



In homework, you proved:

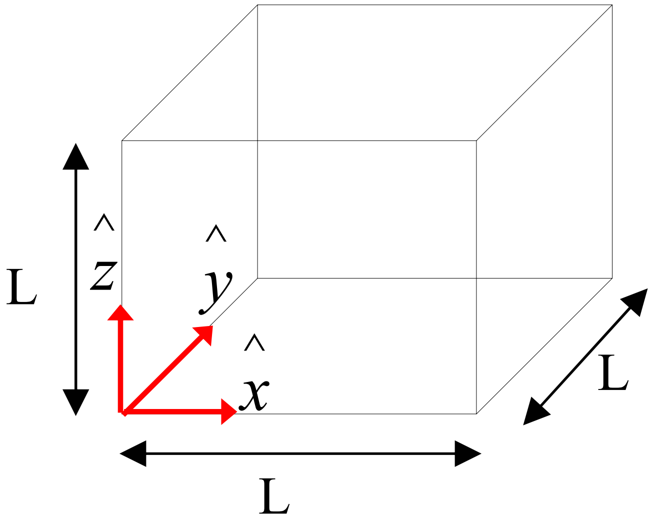
$$\vec{E}_{\|2} = \vec{E}_{\|1} \quad \Rightarrow \boxed{\vec{E}_{\|} = 0}$$

$$\vec{D}_{\perp 1} - \vec{D}_{\perp 2} = \rho_s$$

$$\vec{H}_{\|2} - \vec{H}_{\|1} = \vec{J}_s$$

$$\vec{B}_{\perp 1} = \vec{B}_{\perp 2} \quad \Rightarrow \boxed{\vec{B}_{\perp} = 0}$$

Calculate E(r,t) and B(r,t)



1. We are interested in fields oscillating with frequency ω.
2. We know E,B satisfy wave equation:

$$\nabla^2 \vec{E}(\vec{r},t) = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E}(\vec{r},t) \qquad \nabla^2 \vec{B}(\vec{r},t) = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{B}(\vec{r},t)$$

3. We know plane waves satisfy wave equation.

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right]$$

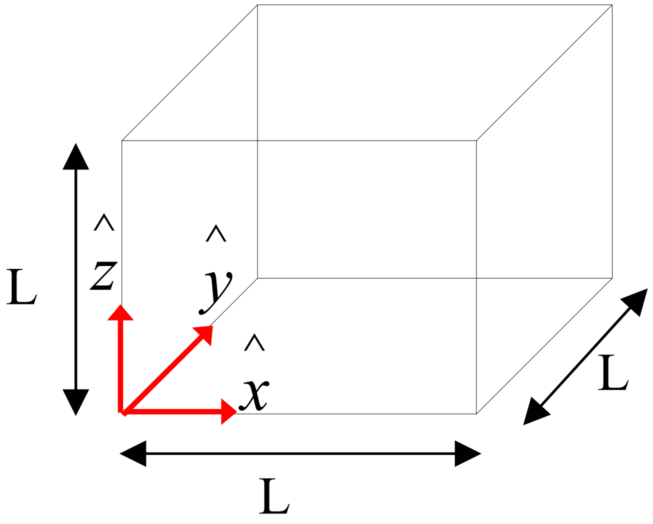
4. What about boundary conditions? A single plane wave will not necessarily obey: $\vec{E}_{\parallel} = 0$

   Let's test for example:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0 \, \hat{z} \, e^{i(k\hat{x}\cdot\vec{r} - \omega t)} \right]$$

   We will check on all six sides of the box.

Boundary conditions:



$$\vec{E}(\vec{r},t) = \text{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} - \omega t)} \right]$$

Boundary conditions:

$$\vec{E}(\vec{r},t) = \text{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} - \omega t)} \right]$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad E_0 = E_{0real} + iE_{0imag}$$

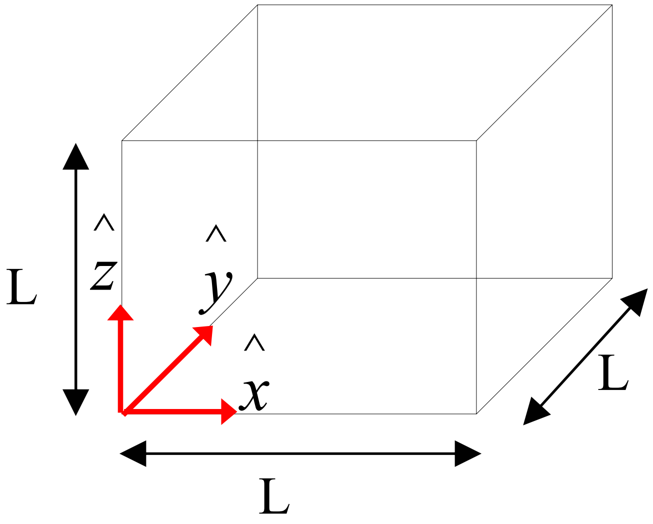
$$\vec{E}(\vec{r},t) = \text{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} - \omega t)} \right]$$

$$= \hat{z}\cdot \text{Re}\left[\left(E_{0real} + iE_{0imag}\right)\cdot\left(\cos(k\hat{x}\cdot\vec{r} - \omega t) + i\sin(k\hat{x}\cdot\vec{r} - \omega t)\right)\right] =$$

$$= \hat{z}\cdot \text{Re}\left[E_{0real}\cos(k\hat{x}\cdot\vec{r} - \omega t) + iE_{0imag}\cos(k\hat{x}\cdot\vec{r} - \omega t) + E_{0real} i\sin(k\hat{x}\cdot\vec{r} - \omega t) - E_{0imag}\sin(k\hat{x}\cdot\vec{r} - \omega t)\right]$$

$$= \hat{z}\cdot\left(E_{0real}\cos(k\hat{x}\cdot\vec{r} - \omega t) - E_{0imag}\sin(k\hat{x}\cdot\vec{r} - \omega t)\right)$$

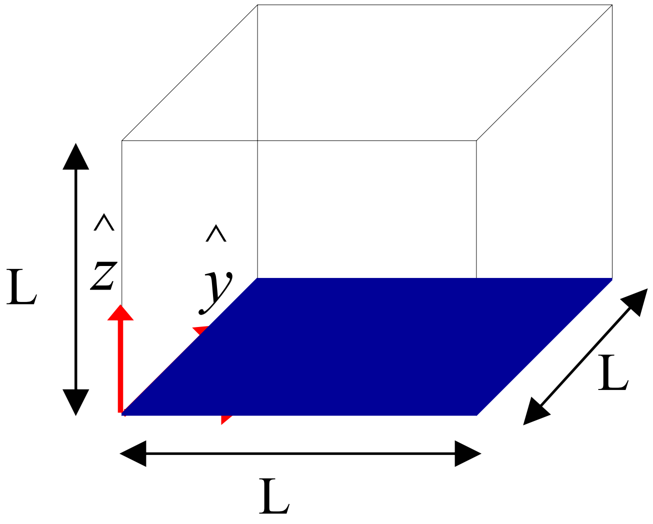
Boundary conditions:



$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} - \omega t)} \right]$$

$$\vec{E}(\vec{r},t) = \hat{z}\cdot\left(E_{0real}\cos(k\hat{x}\cdot\vec{r} - \omega t) - E_{0imag}\sin(k\hat{x}\cdot\vec{r} - \omega t) \right)$$

Boundary conditions:

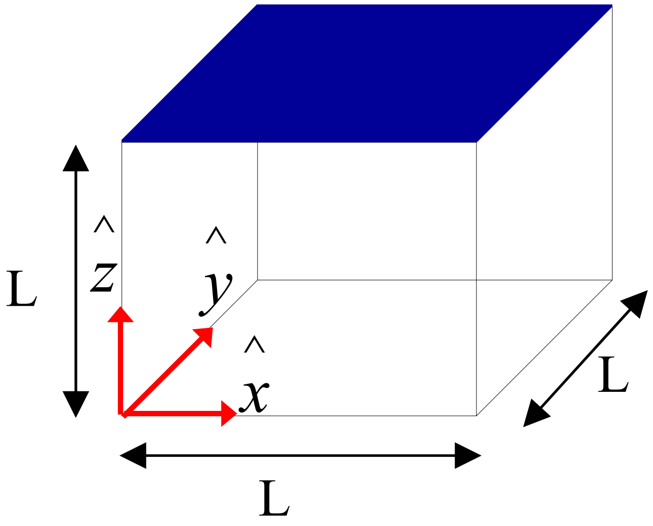


$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} - \omega t)} \right]$$

The plane z=0:

Since E is in the z direction, the component of E parallel to the blue plane is always zero.

Boundary conditions:

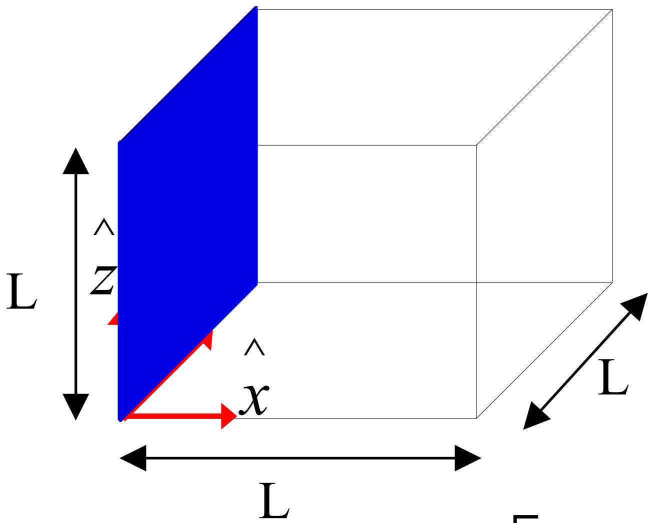


$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0 \, \hat{z} \, e^{i(k\hat{x}\cdot\vec{r} - \omega t)} \right]$$

The plane z=L:

Since E is in the z direction, the component of E parallel to the blue plane is always zero.

Boundary conditions:



$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0 \,\hat{z}\, e^{i(k\hat{x}\cdot\vec{r}-\omega t)} \right]$$

The plane x=0:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0 \,\hat{z}\, e^{i(k\hat{x}\cdot\vec{r}-\omega t)} \right] =$$

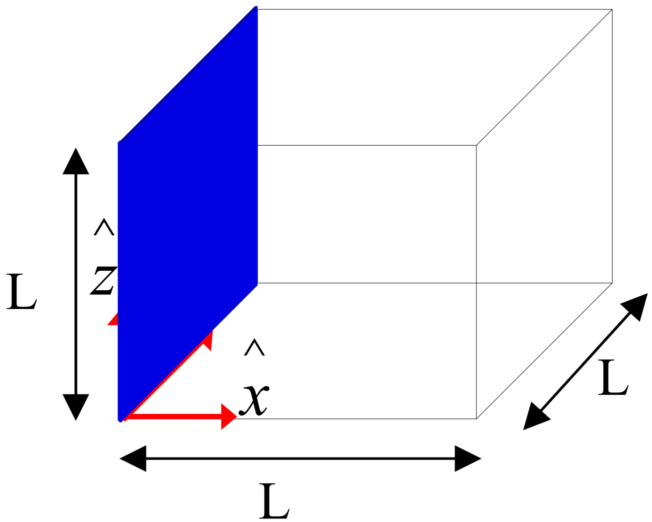
$$= \hat{z}\cdot\left(E_{0real}\cos(k\hat{x}\cdot\vec{r}-\omega t) - E_{0imag}\sin(k\hat{x}\cdot\vec{r}-\omega t) \right)$$

0 0

$$= \hat{z}\cdot\left(E_{0real}\cos(-\omega t) - E_{0imag}\sin(-\omega t) \right)$$

Does not solve boundary condition!!!

Boundary conditions:



Let's try something:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0\, \hat{z}\, e^{i(k\hat{x}\cdot\vec{r}-\omega t)} \right]$$

$$+\,\mathrm{Re}\left[E_0\, \hat{z}\, e^{i(k\hat{x}\cdot\vec{r}+\omega t)} \right]$$

The plane x=0:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0\, \hat{z}\, e^{i(k\hat{x}\cdot\vec{r}-\omega t)} \right] + \mathrm{Re}\left[E_0\, \hat{z}\, e^{i(k\hat{x}\cdot\vec{r}+\omega t)} \right]$$

$$= \left(E_{0real} \cos(k\hat{x}\cdot\vec{r}-\omega t) - E_{0imag} \sin(k\hat{x}\cdot\vec{r}-\omega t) \right)\hat{z} + \left(E_{0real} \cos(k\hat{x}\cdot\vec{r}+\omega t) - E_{0imag} \sin(k\hat{x}\cdot\vec{r}+\omega t) \right)\hat{z}$$

0 0 0 0

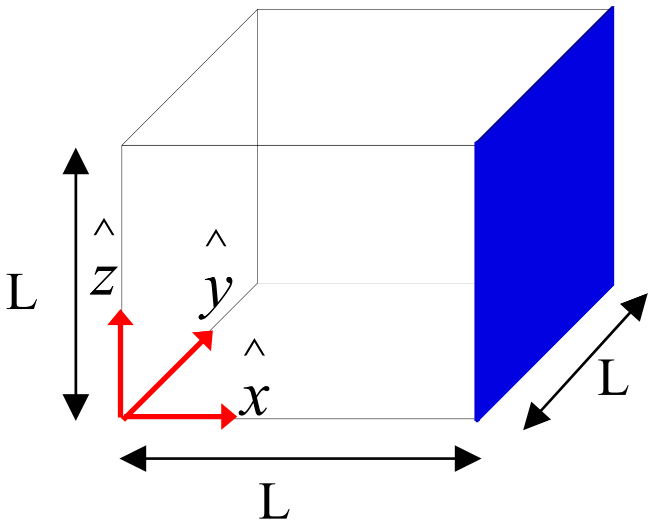
$$= (-E_{0imag})\,\hat{z}\sin(-\omega t) + (-E_{0imag})\,\hat{z}\sin(\omega t) + E_{0real}\cos(-\omega t) + E_{0real}\cos(\omega t)$$

$$= 2E_{0real}\cos(\omega t) = 0 \text{ if and only if } E_{0real} = 0$$

$\sin(\theta) = -\sin(-\theta)$ $\cos(\theta) = \cos(-\theta)$

Does solve boundary condition!!! (At least at x=0). (z=0, L still ok.)

Boundary conditions:



$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} - \omega t)} \right] + \mathrm{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} + \omega t)} \right]$$

The plane x=L:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} - \omega t)} \right] + \mathrm{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} + \omega t)} \right]$$

$$= -E_{0imag} \hat{z} \sin\left(k\hat{x}\cdot\vec{r} - \omega t \right) - E_{0imag} \hat{z} \sin\left(k\hat{x}\cdot\vec{r} + \omega t \right)$$

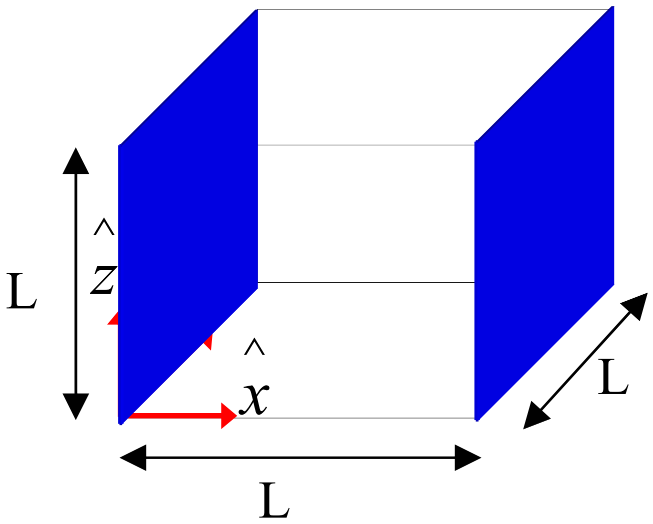
$$= -E_{0imag} \hat{z} \left[\sin(kL - \omega t) + \sin(kL + \omega t) \right]$$

If and only if:

$$= -2E_{0imag} \hat{z} \sin(kL)\cos(\omega t) = 0?$$

$$k_n = n\pi / L$$

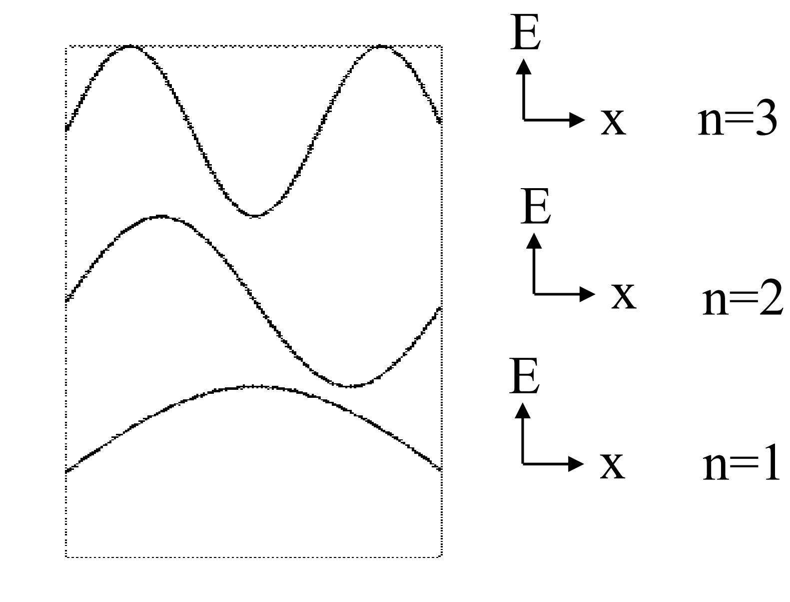
"Standing waves"



$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0\,\hat{z}\,e^{i(k\hat{x}\cdot\vec{r}-\omega t)}\right] + \mathrm{Re}\left[E_0\,\hat{z}\,e^{i(k\hat{x}\cdot\vec{r}+\omega t)}\right]$$

$$= -2E_{0imag}\,\hat{z}\sin(k_n x)\cos(\omega t)$$

$$k_n = n\pi / L$$



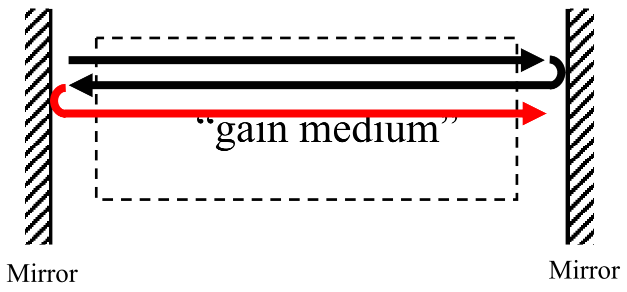
$$\omega_n = ck_n = nc\pi / L$$

Note: amplitude E_0 can have any value.
Later we will find E_0 is quantized itself.

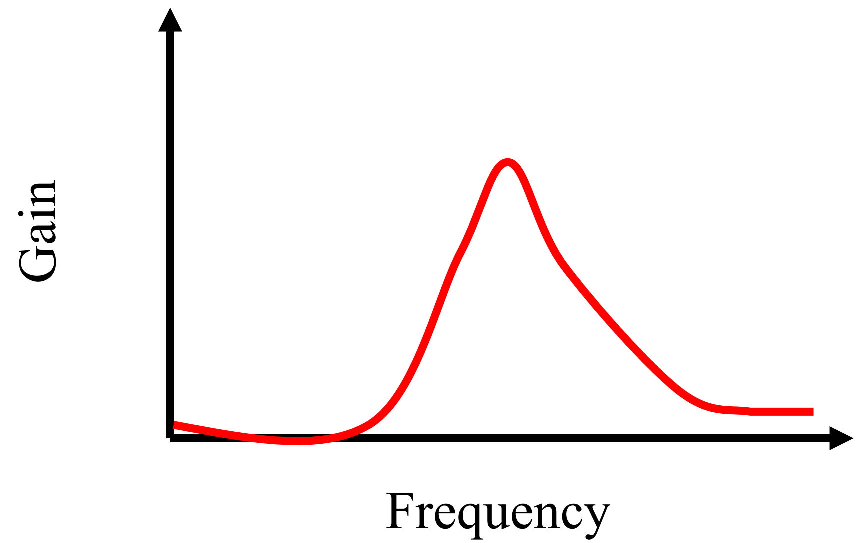
By confining the wave in the direction of travel (the x direction), we have created certain allowed frequencies or "modes".

Later in the lecture we will confine the wave in the direction perpendicular to the direction of travel, such as in a fiber. We will create allowed modes there, too.

"Single vs. multi-mode lasing"

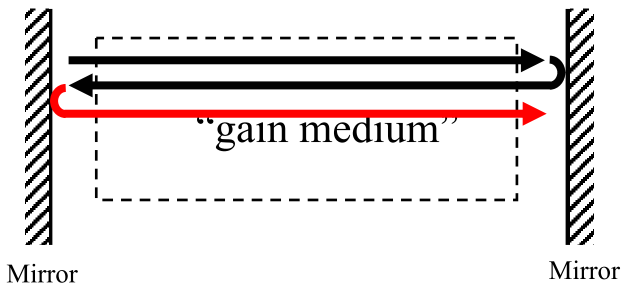


$$\omega_n = ck_n = nc\pi / L$$

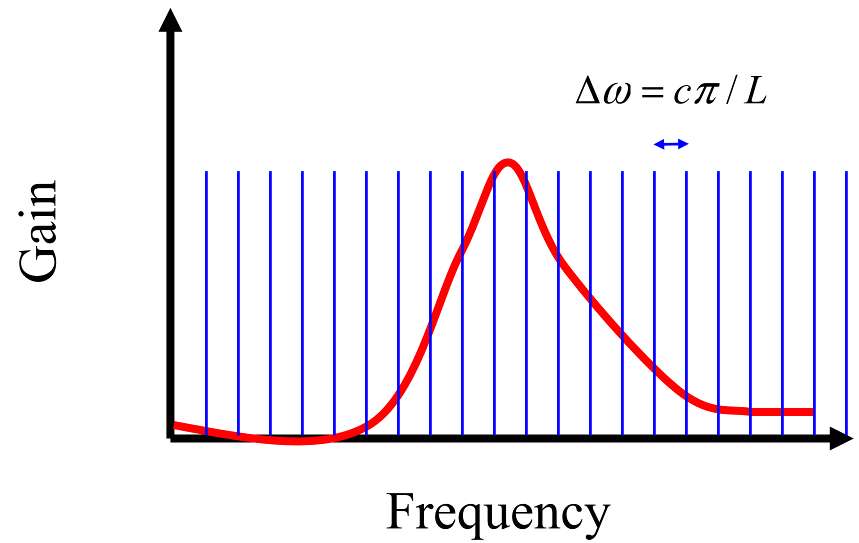


“Single vs. multi-mode lasing”

MULTI-MODE



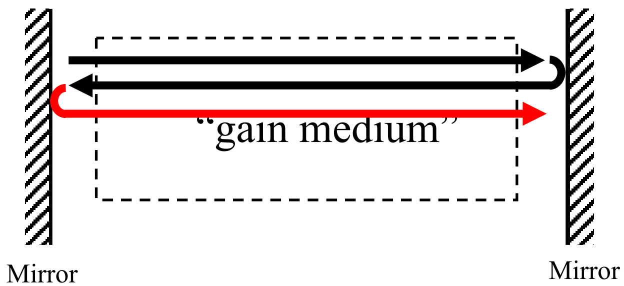
$$\omega_n = ck_n = nc\pi / L$$



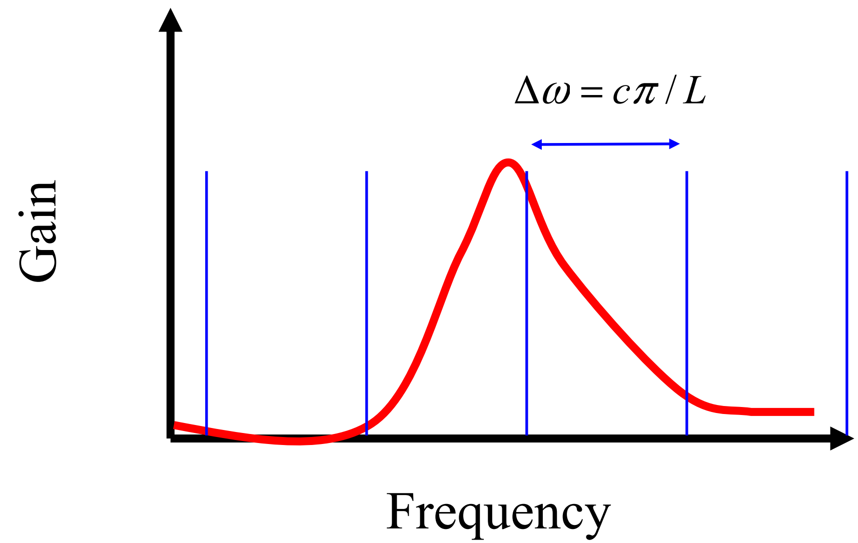
"Single vs. multi-mode lasing"

SINGLE MODE

Obviously, single mode lasers are better for communications.



$$\omega_n = ck_n = nc\pi / L$$

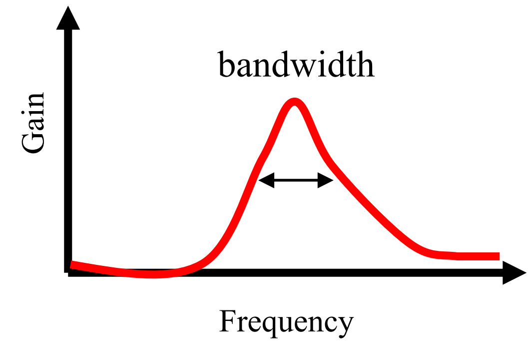


What is the disadvantage of making
L short to go to single mode?

Also discuss mode "width" (see HW#1)

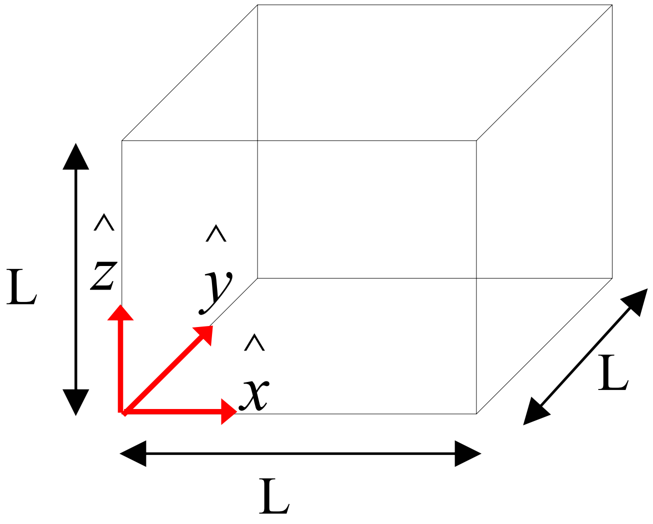
Typical gain bandwidths:

Gain medium	Wavelength	Bandwidth (Hz)
HeNe	6328 Å	$1.5\ 10^9$
Nd:glass	1.06 μm	$3\ 10^{12}$
Nd:YAG	1.06 μm	$120\ 10^9$
Ruby	6943 Å	$6\ 10^{10}$
Argon ions	350 – 520 Å	$3.5\ 10^9$
Ti:Saphire	0.7-1.1 μm	$100\ 10^{12}$
Rh.-6G dye	0.56-0.64 μm	$5\ 10^{12}$
CO_2 gas	10 μm	$60\ 10^6$
Er doped fiber	1.55 μm	$4\ 10^{12}$
Diode lasers	0.7-1.6 μm	10^{13}



Back to the box…

Boundary conditions:

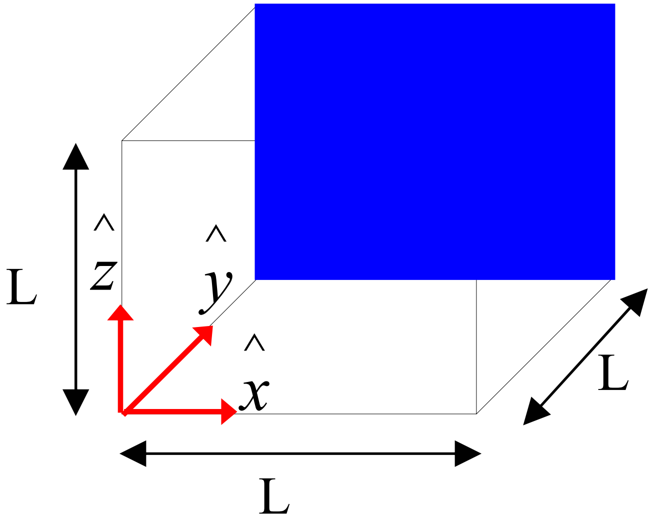


$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} - \omega t)} \right]$$

$$+ \mathrm{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} + \omega t)} \right]$$

$$= 2E_0 \hat{z} \sin(k_n x)\cos(\omega t)$$

Boundary conditions:



$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} - \omega t)} \right] + \mathrm{Re}\left[E_0 \hat{z} e^{i(k\hat{x}\cdot\vec{r} + \omega t)} \right]$$

$$= 2E_0 \hat{z} \sin(k_n x)\cos(\omega t)$$

The plane y=L:

$$\vec{E}(\vec{r},t) = -2E_{0imag} \hat{z} \sin(k_n x)\cos(\omega t) \neq 0$$

Now what?

We need to add a bunch of plane waves together to satisfy all bd. conditions.

Too hard, we'll be here all day. Maybe another time.

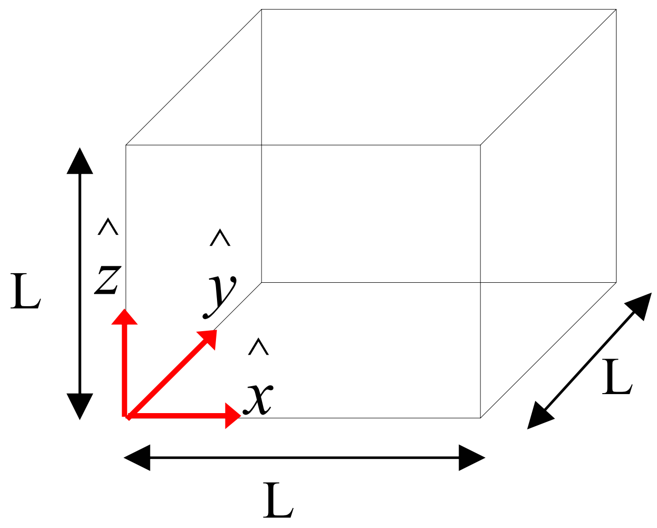
We have learned the hard way that plane waves are not the most efficient solution to every problem.

Especially problems where the fields are confined.

Back to the drawing board…

We have to know, e.g. for fiber optics.

Calculate E(r,t) and B(r,t)



$$\nabla^2 \vec{E}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t) = 0 \qquad \vec{E}_{\parallel} = 0$$

How do we solve this?

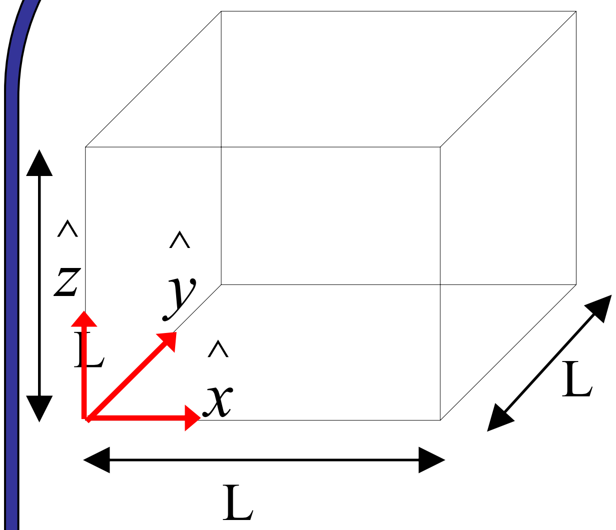
Try: $\vec{E}(\vec{r},t) = \vec{E}(\vec{r})e^{i\omega t}$

Then: $\nabla^2 \vec{E}(\vec{r}) + k_0^2 \vec{E}(\vec{r}) = 0$

So long as: $k_0^2 = \left(\frac{\omega}{c}\right)^2$

Called the *Helmholtz equation.*

Calculate E(r,t) and B(r,t)



$$\nabla^2 \vec{E}(\vec{r}) + k_0^2 \vec{E}(\vec{r}) = 0$$

Work on: $\nabla^2 E_z(x,y,z) + k_0^2 E_z(x,y,z) = 0$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_z(x,y,z) + k_0^2 E_z(x,y,z) = 0$$

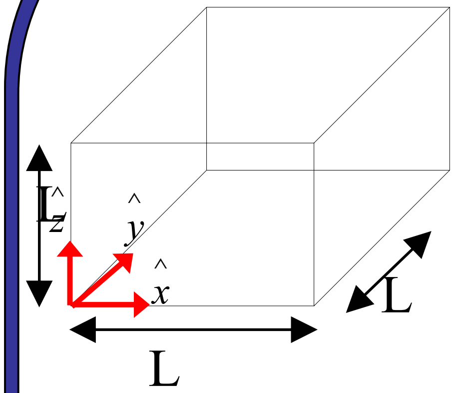
Let: $E_z(x,y,z) = f_1(x) f_2(y) f_3(z)$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f_1(x) f_2(y) f_3(z) + k_0^2 f_1(x) f_2(y) f_3(z) = 0$$

Divide by $f_1(x) f_2(y) f_3(z)$:

$$\frac{1}{f_1(x)} \frac{\partial^2 f_1(x)}{\partial x^2} + \frac{1}{f_2(y)} \frac{\partial^2 f_2(y)}{\partial y^2} + \frac{1}{f_3(z)} \frac{\partial^2 f_3(z)}{\partial z^2} + k_0^2 = 0$$

Calculate E(r,t) and B(r,t)



$$\frac{1}{f_1(x)}\frac{\partial^2 f_1(x)}{\partial x^2}+\frac{1}{f_2(y)}\frac{\partial^2 f_2(y)}{\partial y^2}+\frac{1}{f_3(z)}\frac{\partial^2 f_3(z)}{\partial z^2}+k_0^2=0$$

$$\frac{1}{f_1(x)}\frac{\partial^2 f_1(x)}{\partial x^2}=-\frac{1}{f_2(y)}\frac{\partial^2 f_2(y)}{\partial y^2}-\frac{1}{f_3(z)}\frac{\partial^2 f_3(z)}{\partial z^2}-k_0^2$$

Right hand side constant with respect to x: Call it $-k_x^2$.

$$\frac{1}{f_1(x)}\frac{\partial^2 f_1(x)}{\partial x^2}=-k_x^2$$

$$k_x^2+k_y^2+k_z^2=k_0^2$$

$$f_1(x)=C_1\sin(k_x x)+C_2\cos(k_x x)$$

Similarly,

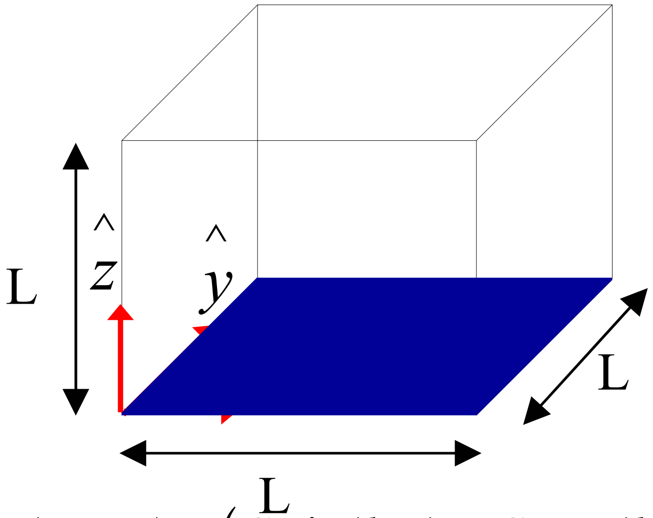
$$f_2(x)=C_3\sin(k_y y)+C_4\cos(k_y y)$$

$$f_3(x)=C_5\sin(k_z z)+C_6\cos(k_z z)$$

$$E_z(x,y,z)=f_1(x)f_2(y)f_3(z)$$

$$E_z(x,y,z) = \left(C_1 \sin(k_x x) + C_2 \cos(k_x x)\right) \cdot \left(C_3 \sin(k_y y) + C_4 \cos(k_y y)\right) \cdot \left(C_5 \sin(k_z z) + C_6 \cos(k_z z)\right)$$

Boundary conditions:

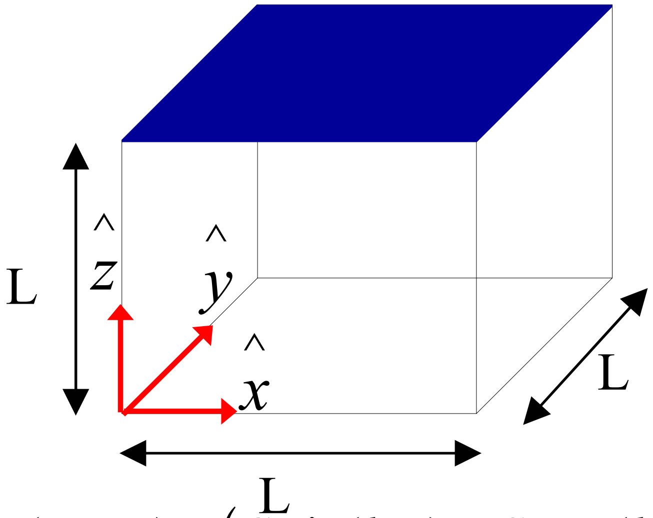


$$E_z(x,y,z) = \left(C_1 \sin(k_x x) + C_2 \cos(k_x x)\right)\cdot\left(C_3 \sin(k_y y) + C_4 \cos(k_y y)\right)\cdot\left(C_5 \sin(k_z z) + C_6 \cos(k_z z)\right)$$

The plane z=0:

Since we are considering E_z, the component of E parallel to the blue plane is always zero.

Boundary conditions:

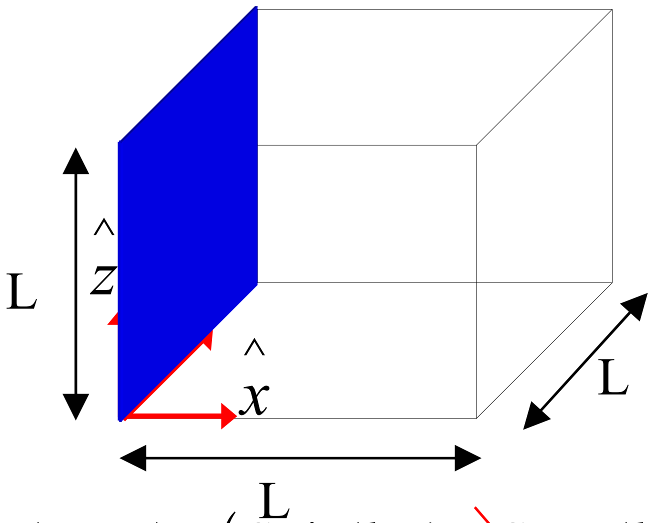


$$E_z(x,y,z) = \left(C_1 \sin(k_x x) + C_2 \cos(k_x x)\right) \cdot \left(C_3 \sin(k_y y) + C_4 \cos(k_y y)\right) \cdot \left(C_5 \sin(k_z z) + C_6 \cos(k_z z)\right)$$

The plane z=L:

Since we are considering E_z, the component of E parallel to the blue plane is always zero.

Boundary conditions:

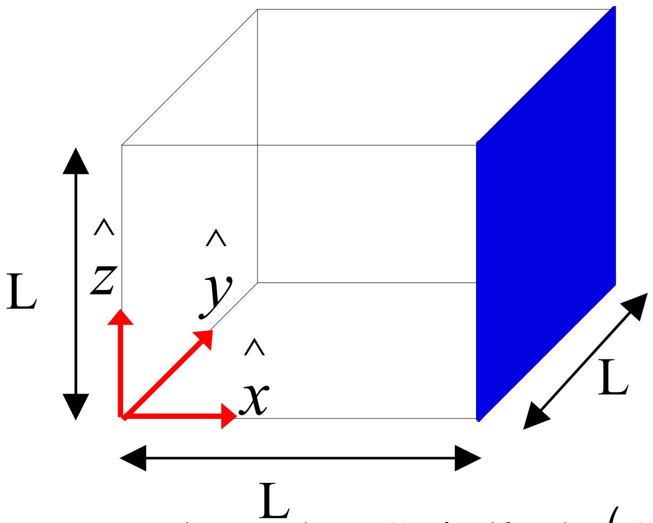


$$E_z(x,y,z) = \left(C_1 \sin(k_x x) + C_2 \cos(k_x x)\right)\cdot\left(C_3 \sin(k_y y) + C_4 \cos(k_y y)\right)\cdot\left(C_5 \sin(k_z z) + C_6 \cos(k_z z)\right)$$

0

The plane x=0:
=> C_2=0

Boundary conditions:

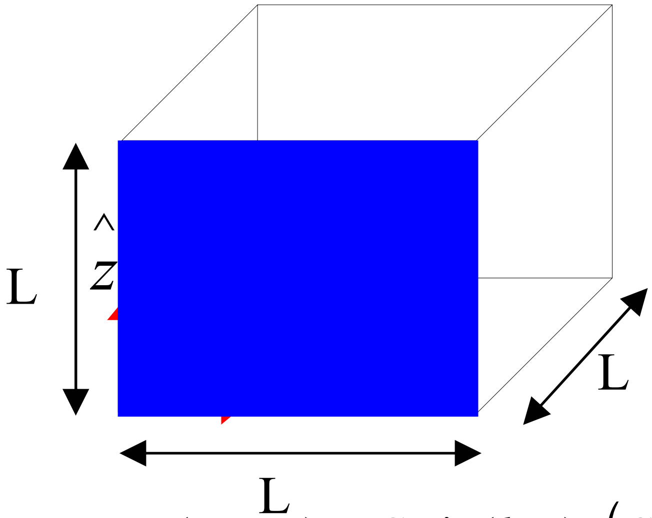


$$E_z(x,y,z) = C_1 \sin(k_x x)\cdot\left(C_3 \sin(k_y y) + C_4 \cos(k_y y)\right)\cdot\left(C_5 \sin(k_z z) + C_6 \cos(k_z z)\right)$$

The plane x=L:

=> $k_x = n_x \pi/L$

Boundary conditions:



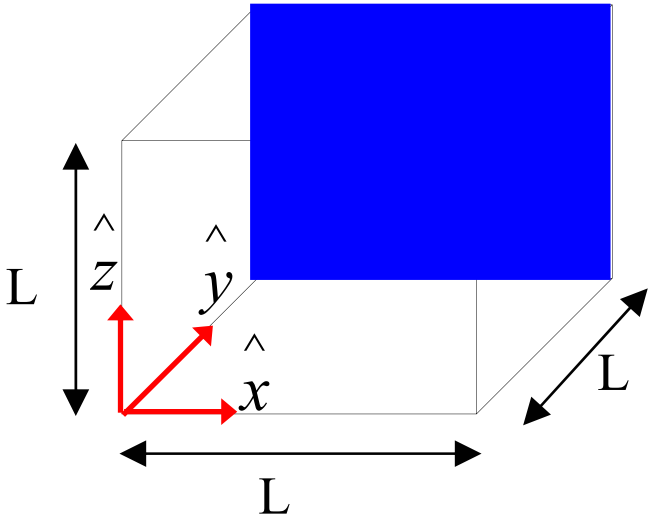
$$E_z(x,y,z) = C_1 \sin(k_x x)\cdot\left(C_3 \sin(k_y y) + C_4 \cos(k_y y)\right)\cdot\left(C_5 \sin(k_z z) + C_6 \cos(k_z z)\right)$$

0

The plane y=0:

=> $C_4 = 0$

Boundary conditions:

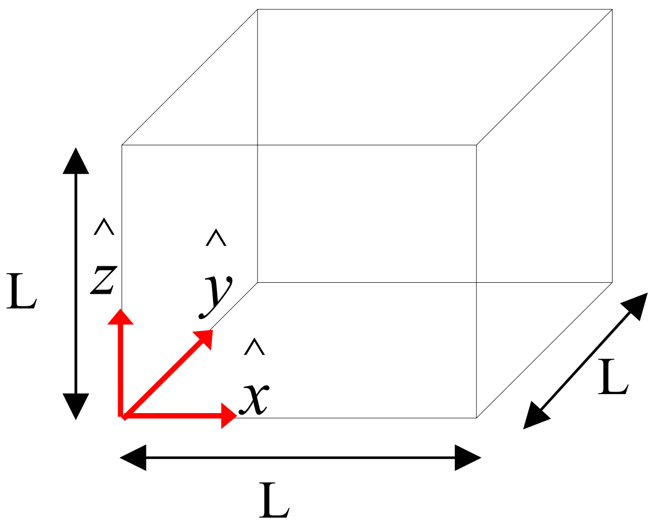


$$E_z(x,y,z) = C_1 \sin(k_x x) \cdot \sin(k_y y) \cdot \left(C_5 \sin(k_z z) + C_6 \cos(k_z z)\right)$$

The plane y=L:

=> $k_x = n_x \pi/L$

Boundary conditions:



We can do the same for E_x, E_y:

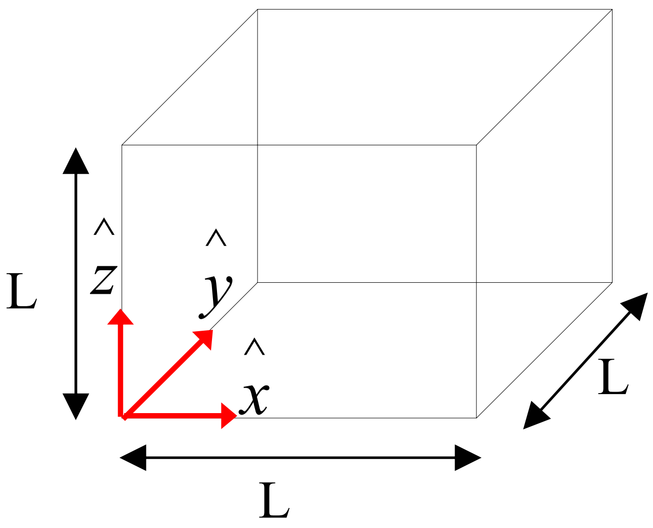
$$E_z(x,y,z) = C_1 \sin(k_x x)\cdot \sin(k_y y)\cdot \left(C_5 \sin(k_z z) + C_6 \cos(k_z z)\right)$$

$$E_y(x,y,z) = C_7 \sin(k_x x)\cdot \left(C_8 \sin(k_y y) + C_9 \cos(k_y y)\right)\cdot \sin(k_z z)$$

$$E_x(x,y,z) = \left(C_{10} \sin(k_x x) + C_{11} \cos(k_x x)\right)\cdot \sin(k_y y)\cdot \sin(k_z z)$$

This obeys the Helmholtz equation (which is equivalent to the wave equation), and the boundary conditions. But we still need to check Maxwell's equations. In particular, $\vec{\nabla}\cdot \vec{E} = 0$

Boundary conditions:



If we implement $\vec{\nabla} \cdot \vec{E} = 0$ we find two things:

First:

$$E_x(x,y,z,t) = E_1 \cos(k_x x) \cdot \sin(k_y y) \cdot \sin(k_z z) e^{i\omega t}$$

$$E_y(x,y,z,t) = E_2 \sin(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) e^{i\omega t}$$

$$E_z(x,y,z,t) = E_3 \sin(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) e^{i\omega t}$$

Second:

$$E_1 k_x + E_2 k_y + E_3 k_z = 0$$

$$\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k_z^2 = \left(\frac{\pi}{L}\right)^2 \left(n_x^2 + n_y^2 + n_z^2\right)$$

n_1, n_2, n_3 integers; at least two must be non-zero.

Each combination of n_1, n_2, n_3 is a "mode".
Discuss superposition.
The magnitude of E_1, E_2, E_3 still can be any value!
(Subject to constraints above.)

What was the point?

- First, we got introduced to standing waves. (We now know the difference between single and multimode lasing)
- Second, we got introduced to simplest form of a confined E-M field: a BOX.
- Later we will do a CYLINDER (e.g. a fiber)
- Then we will do a Gaussian beam.

How much energy is in the box?

Instantaneous energy per unit volume:

$$u(\vec{r},t) = \frac{1}{2}\left(\varepsilon_0\left[\vec{E}(\vec{r},t)\right]^2 + \mu_0\left[\vec{H}(\vec{r},t)\right]^2\right)$$

Total energy in box:

$$U_{total} = \iiint_{box} u(\vec{r},t)\,dV$$

It can be shown that for the box:

$$U_{total} = \frac{1}{2}\left(n_x^2 + n_y^2 + n_z^2\right)\left(E_1^2 + E_2^2 + E_3^2\right)$$

So, the amount of energy in the box can have any value.
We will show that this leads to a problem and must be wrong.
The energy in the box must be quantized: these are *photons*.

Conclusion

- Lecture 1: Introduction
- Lecture 2: Plane waves (unconfined)
- Lecture 3: Box (simplest confined case)
- Lecture 4: We will consider energy in the box.