Lecture 2: Review of classical electromagnetic waves

Outline for today: Classical E/M "Bootcamp"

- Maxwell's equations
- Derivation of the wave equation
- Boundary conditions
- Reflection and refraction
- Ray tracing and conditions of applicability
- Lenses

Reading: Verdeyen chapter 1, inside cover, and your prerequisite electromagnetics course.

Maxwell's equations in vacuum (Note: Verdeyen left out two equations!)

$$\vec{\nabla} \cdot \vec{e} = \rho / \varepsilon_0 \qquad \vec{\nabla} \cdot \vec{b} = 0$$

$$\vec{\nabla}\, x\, \vec{e} = -\frac{\partial \vec{b}}{\partial t} \qquad \vec{\nabla}\, x\, \vec{b} = \mu_0 \varepsilon_0 \frac{\partial \vec{e}}{\partial t} + \mu_0 \vec{j}$$

$$\vec{F} = q(\vec{e} + \vec{v}\, x\, \vec{b})$$

$$\varepsilon_0 = 8.85 \times 10^{-12} F / m$$

$$\mu_0 = 4\pi \times 10^{-7} H / m$$

Maxwell's equations: Physical significance

- Particles treated classically: ρ, j
- Fields treated classically: e,b
- Helmholtz theorem:

ANY vector field can be reconstructed from its "div" and "curl"

I.E. if you (the engineer) can set up ρ, j, you can set up e,b !!!

Maxwell's equations: Physical shortcomings

- Matter is not classical, it is quantum
- Matter can only have certain energy levels
- e,b are not classical, they are quantum
- e,b can only have certain energies:
  $E = n\ h\ f$
  - n=number of "photon"
  - h is Planck's constant
  - f is the frequency

What if there are no sources?

$\rho, j=0$

Does that mean e,b=0?

NO!

We can have *waves*.

Why do I care?

Light is a wave.

Wave equation, no sources:

$$\vec{\nabla} x \left(\vec{\nabla} x \vec{e} \right) = \vec{\nabla} x \left(-\frac{\partial \vec{b}}{\partial t} \right) = -\frac{\partial}{\partial t} \left(\vec{\nabla} x \vec{b} \right) = -\frac{\partial}{\partial t} \left(\mu_0 \varepsilon_0 \frac{\partial \vec{e}}{\partial t} \right) = -\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{e}$$

$$\vec{\nabla} x \left(\vec{\nabla} x \vec{e} \right) = \vec{\nabla} \left(\vec{\nabla} \cdot \vec{e} \right) - \nabla^2 \vec{e} = -\nabla^2 \vec{e}$$

$$\nabla^2 \vec{e} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{e}$$

Wave equation, no sources:

$$\nabla^2 \vec{e} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{e}$$

$$\nabla^2 \vec{b} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{b}$$

(Homework problem)

Plane waves (draw picture on board):

$$\vec{e}(\vec{r},t) = \text{Re}\left\{\left[\vec{E}(\omega,\vec{k}_0)\right]e^{j(\omega t-\vec{k}_0\cdot\vec{r})}\right\}$$

$$\vec{b}(\vec{r},t) = \text{Re}\left\{\left[\vec{B}(\omega,\vec{k}_0)\right]e^{j(\omega t-\vec{k}_0\cdot\vec{r})}\right\}$$

is a general solution, provided

$$\omega = \frac{1}{\sqrt{\mu_0\varepsilon_0}}k_0 \qquad \frac{|E|}{|B|} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Defines the speed of light c:

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8 m/s$$

Superpositions of plane waves:

$$\vec{e}_1(\vec{r},t) = \text{Re}\left\{\left[\vec{E}_1(\omega_1,\vec{k}_{01})\right]e^{j(\omega_1 t - \vec{k}_{01}\cdot\vec{r})}\right\}$$

$$\vec{e}_2(\vec{r},t) = \text{Re}\left\{\left[\vec{E}_2(\omega_2,\vec{k}_{02})\right]e^{j(\omega_2 t - \vec{k}_{02}\cdot\vec{r})}\right\}$$

$$A\,\vec{e}_1(\vec{r},t) + B\,\vec{e}_2(\vec{r},t)$$

Is also a solution.
In general:

$$\vec{e}(\vec{r},t) = \sum_n A_n\,\vec{e}_n(\vec{r},t)$$

Polarization:

$$\vec{e}(\vec{r},t) = \mathrm{Re}\left\{ \left[\vec{E}(\omega,\vec{k}_0) \right] e^{j(\omega t - \vec{k}_0 \cdot \vec{r})} \right\}$$

$$\vec{B}(\omega,\vec{k}_0) = \frac{1}{\omega} \vec{k}_0 \times \vec{E}(\omega,\vec{k}_0)$$

Draw on board: linear, circular polarization.

Poynting vector

$$\vec{S} = c^2 \varepsilon_0 \vec{E} \, x \, \vec{B} = \vec{E} \, x \, \vec{H} \qquad \text{Watts/m}^2$$

We are most interested in time average:

$$< S >_T = \frac{1}{2} c^2 \varepsilon_0 E^2 \qquad \text{Watts/m}^2$$

Called ***intensity*** or ***irradiance***:

the average energy per unit time per unit area

$$< S >_T \cdot Area$$

Called ***optical power*** or ***radiant flux***

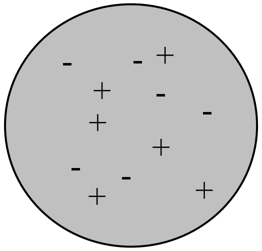
Maxwell's equations in matter

- Why do I care?
- Light passes through matter (glass, etc)
- We need to understand how it propagates in matter.

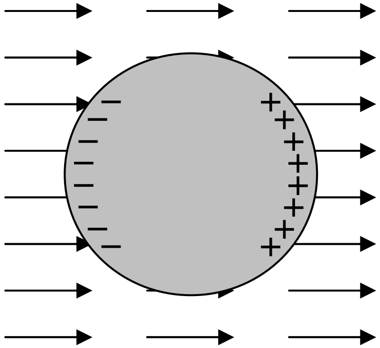
Electric dipole moment

$$\vec{p} = \sum_{i=1}^{N} q_i \vec{r}_i$$

For neutral matter,
usually 0:



Polarization:
Electric field induces dipole moment:



$$\vec{P} = \chi_e \varepsilon_0 \vec{E}$$

P is dipole moment per unit volume.
This equation defines χ_e
The e is for electric.
Verdeyen does not use the e.
Also works for finite ω.

Polarization:

$$\vec{P} = \chi \varepsilon_0 \vec{E}$$

We define displacement vector D:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$= \varepsilon_0 \vec{E} + \chi \varepsilon_0 \vec{E}$$

$$= \varepsilon_0 (1 + \chi) \vec{E}$$

$$= \varepsilon_0 \varepsilon_\tau \vec{E}$$

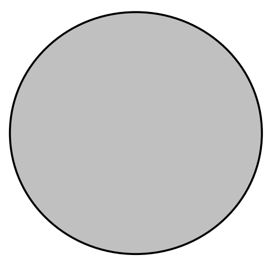
Index of refraction: $n = \sqrt{\varepsilon_\tau}$

Wave velocity: v = c/n

Magnetic dipole moment

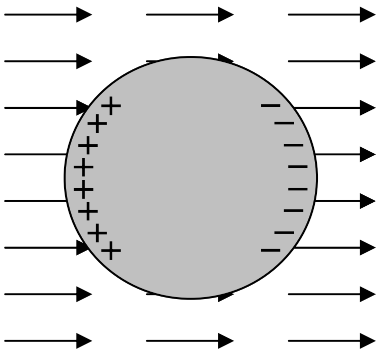
$$\vec{m} = \frac{1}{2}\int_V \vec{r}\, x\, \vec{J}(\vec{r})\, dV$$

For neutral matter,
usually 0:



Magnetization:

Magnetic field induces magnetic dipole moment:



$$\vec{M} = \frac{\chi_m}{\mu_0} \vec{B}$$

M is magnetic dipole moment per unit volume.

This equation defines χ_m

(Not in Verdeyen)

Magnetization:

$$\vec{M} = \frac{\chi_m}{\mu_0}\vec{B}$$

We define field H:

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$= \frac{1}{\mu_0}\vec{B} - \frac{\chi_m}{\mu_0}\vec{B}$$

$$= \vec{B}\frac{1}{\mu_0}(1 - \chi_m) \quad = \vec{B}\frac{1}{\mu_\tau \mu_0}$$

Index of refraction: $n = \sqrt{\varepsilon_\tau \mu_\tau}$

Wave velocity: v = c/n

Maxwell's equations in solids

$$\vec{\nabla} \cdot \vec{d} = \rho_f \qquad \vec{\nabla} \cdot \vec{b} = 0$$

$$\vec{\nabla} x \vec{e} = -\frac{\partial \vec{b}}{\partial t} \qquad \vec{\nabla} x \vec{h} = \frac{\partial \vec{d}}{\partial t} + \vec{j}_f$$

The subscript f is for "free", i.e. those charges that do not participate in the magnetic or electric susceptibility.

Some people think this form of Maxwell's equations is "simpler".

Characteristic impedance:

In vacuum:

$$\frac{|E|}{|B|} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \qquad \frac{|E|}{|H|} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = Z_0 = 377\Omega$$

In dielectric:

$$\frac{|E|}{|B|} = \frac{1}{\sqrt{\mu_0 \mu_\tau \varepsilon_0 \varepsilon_\tau}} = \frac{c}{n} \qquad \frac{|E|}{|H|} = \sqrt{\frac{\mu_0 \mu_\tau}{\varepsilon_0 \varepsilon_\tau}} = Z$$

These are important in antenna and transmission line problems where we want to impedance match a circuit to a plane wave.

Boundary conditions:

$$\vec{a}_n \times (\vec{E}_1 - \vec{E}_2) = 0 \quad \Leftrightarrow \quad \vec{E}_{\|2} = \vec{E}_{\|1}$$

$$\vec{a}_n \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \quad \Leftrightarrow \quad \vec{D}_{\perp 1} - \vec{D}_{\perp 2} = \rho_s$$

$$\vec{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \quad \Leftrightarrow \quad \vec{H}_{\|2} - \vec{H}_{\|1} = \vec{J}_s$$

$$\vec{a}_n \cdot (\vec{B}_1 - \vec{B}_2) = 0 \quad \Leftrightarrow \quad \vec{B}_{\perp 1} = \vec{B}_{\perp 2}$$

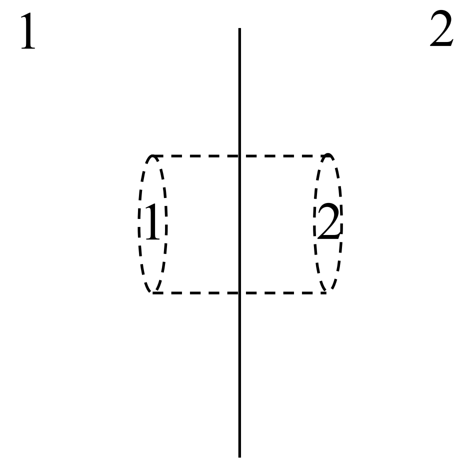
•You will prove in HW, starting from Maxwell's equations

Why do I care?

Important to understand reflection and transmission of light.

Example proof:

$$\vec{a}_n \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$$



We will prove for ρ_s=0.

$$\vec{\nabla} \cdot \vec{d} = 0$$

$$\iiint_V dV \, \vec{\nabla} \cdot \vec{d} = \oint\!\!\oint_S dA \, \vec{d} \cdot \vec{a}_n$$

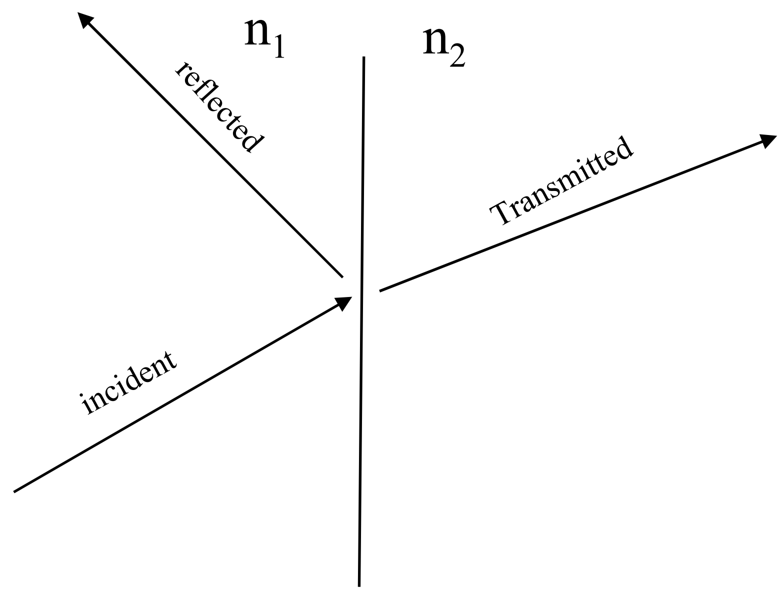
||

0

$$= \vec{d}_1 \cdot \Delta \vec{a}_1 + \vec{d}_2 \cdot \Delta \vec{a}_2 + circ.term$$

$$= \vec{d}_1 \cdot \Delta \vec{a}_1 + \vec{d}_2 \cdot (-\Delta \vec{a}_1) = \vec{a}_n \cdot (\vec{d}_1 - \vec{d}_2)$$

Reflection and refraction:

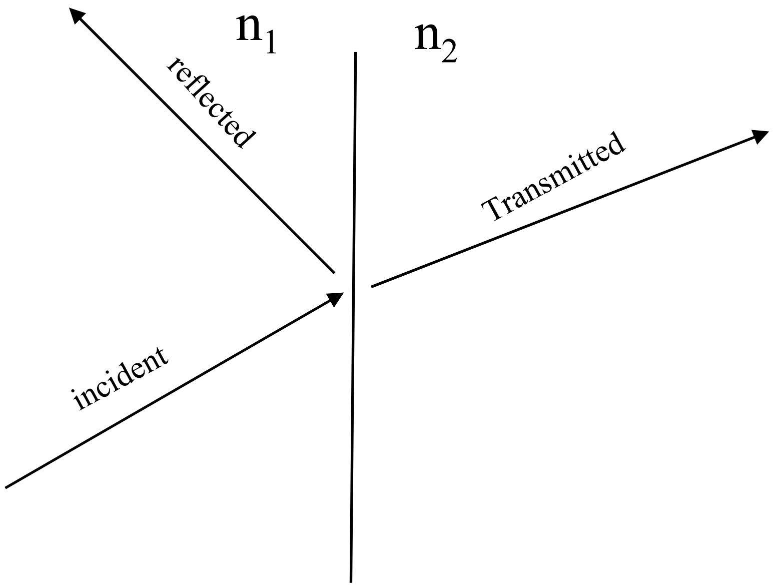


Why do I care?

This is the simplest geometry for reflection and transmission.

This will form the basis of understanding mirrors that form a laser cavity.

Reflection and refraction:



$$\vec{E}_i = \vec{E}_{0i}\, e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

$$\vec{E}_t = \vec{E}_{0t}\, e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

$$\vec{E}_r = \vec{E}_{0r}\, e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

(Write on board)

Reflection and refraction:

$$k_i^2 = \left(\frac{\omega_i}{\mathrm{v}_1}\right)^2 = \left(\frac{n_1\omega_i}{c}\right)^2$$

$$k_t^2 = \left(\frac{\omega_t}{\mathrm{v}_2}\right)^2 = \left(\frac{n_2\omega_t}{c}\right)^2$$

$$k_r^2 = \left(\frac{\omega_r}{\mathrm{v}_1}\right)^2 = \left(\frac{n_1\omega_r}{c}\right)^2$$

Boundary condition:
$E_{||}$ is continuous

$$\left[\vec{E}_{0i}\, e^{i(\vec{k}_i\cdot\vec{r}_B-\omega t)} + \vec{E}_{0r}\, e^{i(\vec{k}_r\cdot\vec{r}_B-\omega t)}\right]_{||} = \left[\vec{E}_{0t}\, e^{i(\vec{k}_t\cdot\vec{r}_B-\omega t)}\right]_{||}$$

So: $\omega_i = \omega_r = \omega_t = \omega$

$$k_i^2 = k_r^2 = \left(\frac{n_1\omega}{c}\right)^2 = k_1^2 \qquad k_t^2 = \left(\frac{n_2\omega}{c}\right)^2 = k_2^2$$

What about direction of k?

$$\vec{k}_i \cdot \vec{r}_B = \vec{k}_t \cdot \vec{r}_B = \vec{k}_r \cdot \vec{r}_B$$

We will explore these equations one by one.

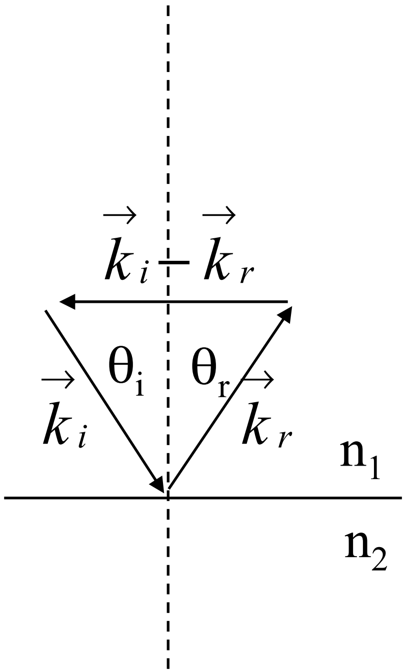
What about direction of k?

$$\left(\vec{k}_i - \vec{k}_r\right)\cdot \vec{r}_B = 0$$

Let the origin be on the surface.

Then the direction of r_B is always parallel to the surface.

(Draw dot product geometry on board).

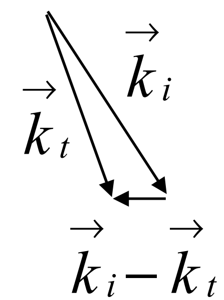


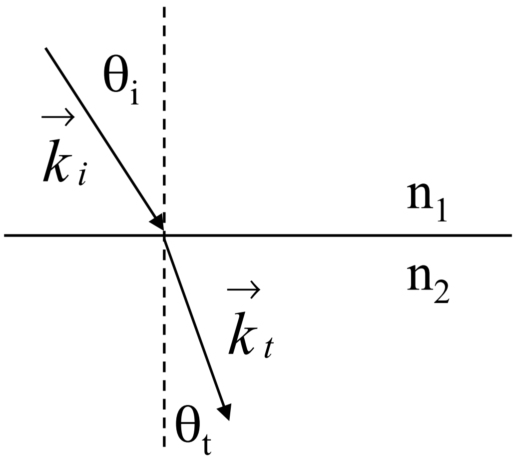
Then this says the component of $\left(\vec{k}_i - \vec{k}_r\right)$ perpendicular to the surface is zero.

$$\theta_i = \theta_r$$

What about direction of k?

$$\left(\vec{k}_i - \vec{k}_t\right) \cdot \vec{r}_B = 0$$



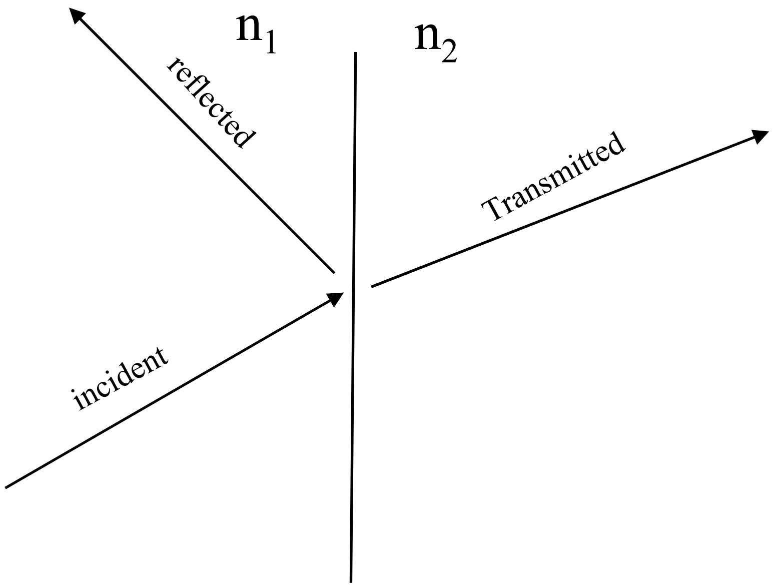


$$k_i \sin(\theta_i) = k_t \sin(\theta_t)$$

$$k_i = (n_1\omega / c) \qquad k_t = (n_2\omega / c)$$

$$n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$$ **Snell's law**

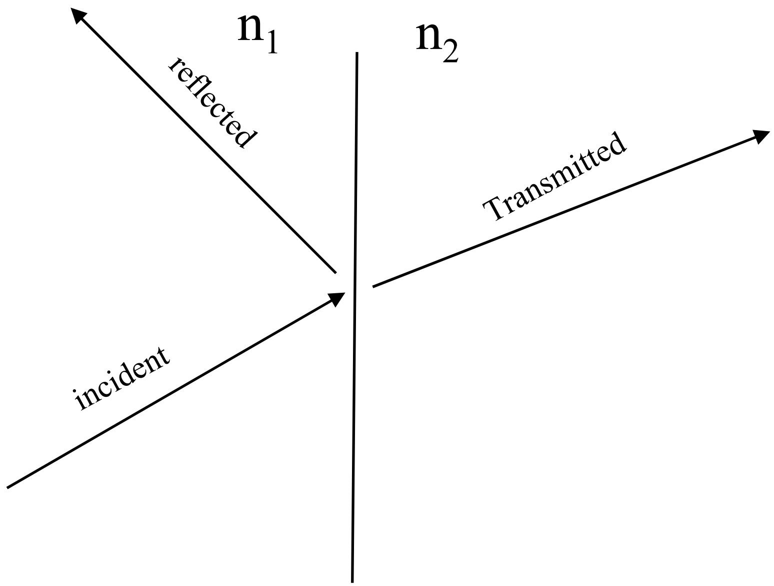
Reflection and refraction:



Now we know about direction of reflected and transmitted waves.

What about the amplitude?

Reflection and refraction:



$$\vec{E}_i = \vec{E}_{0i}\, e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

$$\vec{E}_r = \vec{E}_{0r}\, e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

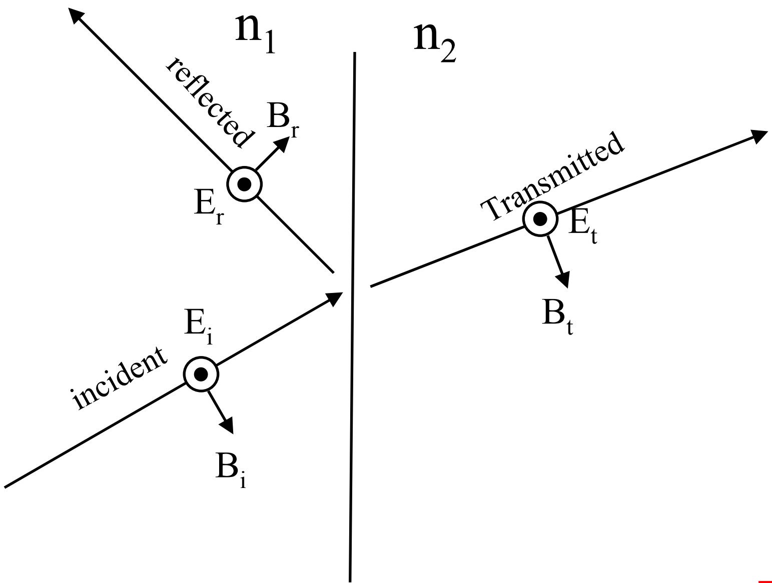
$$\vec{E}_t = \vec{E}_{0t}\, e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

We want to know:

$$\frac{E_r}{E_i} = ??? \qquad \frac{E_t}{E_i} = ???$$

Results are the ***Fresnel Equations.***

Fresnel equations: $E_\perp$



$$\vec{E}_i = \vec{E}_{0i}\, e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

$$\vec{E}_r = \vec{E}_{0r}\, e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

$$\vec{E}_t = \vec{E}_{0t}\, e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

Continuity in $E_{\|}$ =>

$$E_{0i} + E_{0r} = E_{0t}$$

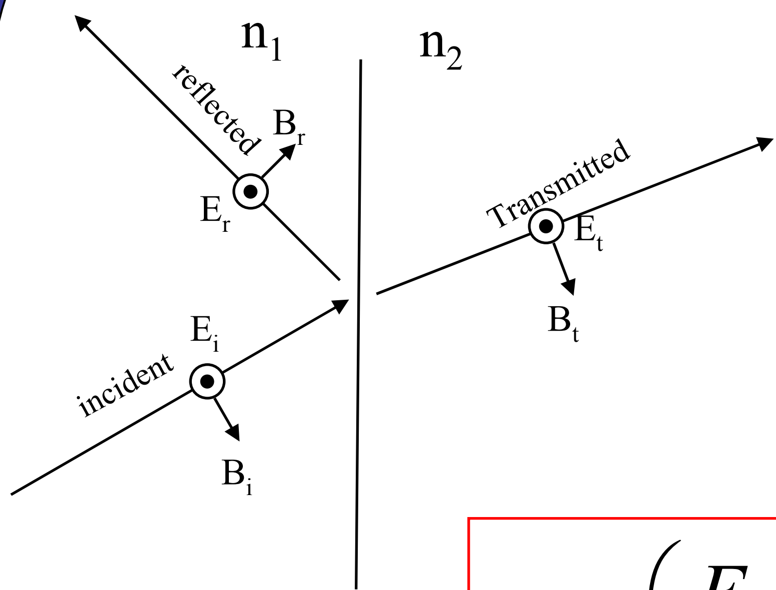
Continuity in $H_{\|}$ =>

$$(H_{0i})_{\|} + (H_{0r})_{\|} = (H_{0t})_{\|}$$

$$-\frac{B_{0i}}{\mu_1}\cos(\theta_i) + -\frac{B_{0r}}{\mu_1}\cos(\theta_r) = \frac{B_{0t}}{\mu_2}\cos(\theta_t)$$

$$-\frac{E_{0i} n_1}{c\mu_1}\cos(\theta_i) + -\frac{E_{0r} n_1}{c\mu_1}\cos(\theta_r) = \frac{E_{0t} n_2}{c\mu_2}\cos(\theta_t)$$

Fresnel equations: $E_\perp$



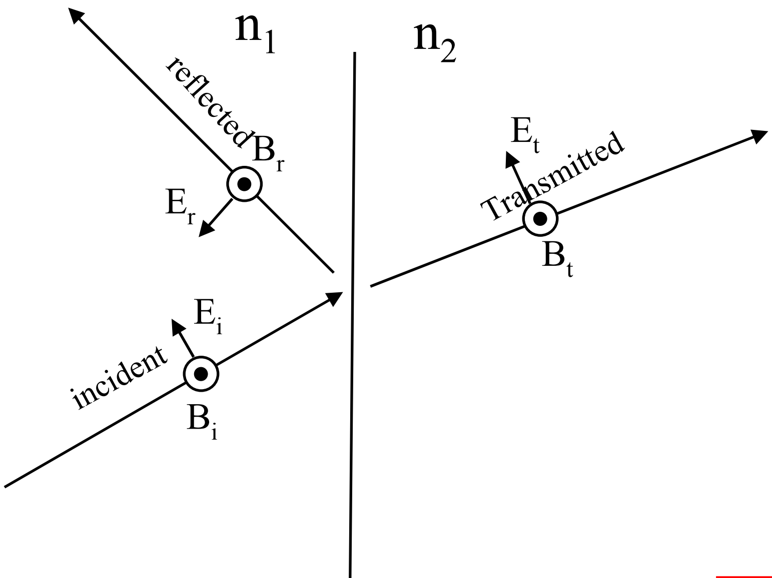
Amplitude reflection coefficients:

$$r_\perp \equiv \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

$$t_\perp \equiv \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

(Write normal incidence formula on the board.)

Fresnel equations: $E_{\parallel}$



$$\vec{E}_i = \vec{E}_{0i}\, e^{i(\vec{k}_i\cdot\vec{r}-\omega t)}$$

$$\vec{E}_r = \vec{E}_{0r}\, e^{i(\vec{k}_r\cdot\vec{r}-\omega t)}$$

$$\vec{E}_t = \vec{E}_{0t}\, e^{i(\vec{k}_t\cdot\vec{r}-\omega t)}$$

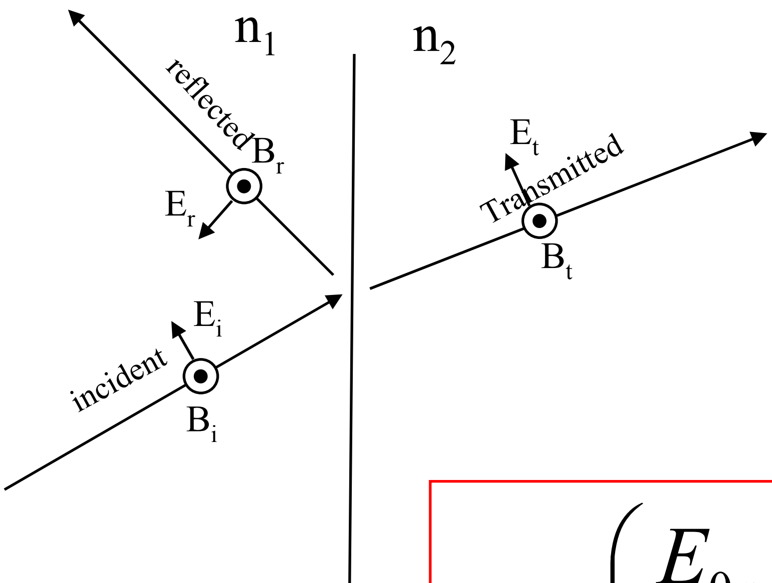
Continuity in $E_{\parallel}$ =>

$$E_{0i}\cos\theta_i - E_{0r}\cos\theta_r = E_{0t}\cos\theta_t$$

Continuity in $H_{\parallel}$ =>

$$\frac{n_1}{\mu_1 c}E_{0i} + \frac{n_1}{\mu_1 c}E_{0r} = \frac{n_2}{\mu 2c}E_{0t}$$

Fresnel equations: $E_{\|}$



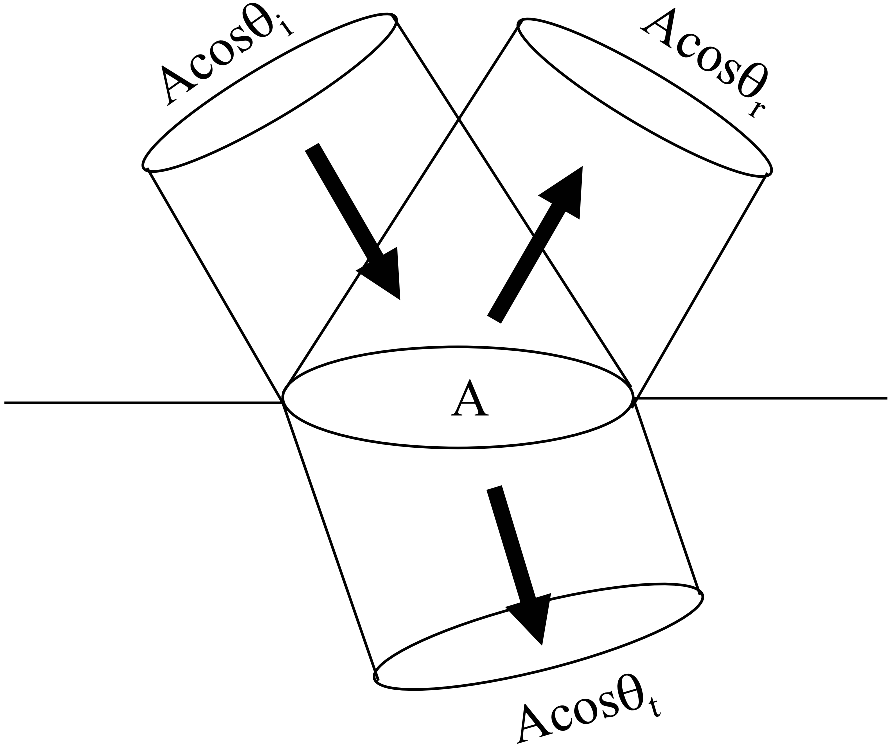
Amplitude reflection coefficients:

$$r_{\|} \equiv \left(\frac{E_{0r}}{E_{0i}} \right)_{\|} = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2}$$

$$t_{\|} \equiv \left(\frac{E_{0r}}{E_{0i}} \right)_{\|} = \frac{2 n_1 \cos\theta_1}{n_2 \cos\theta_1 + n_1 \cos\theta_2}$$

(Write normal incidence formula on the board.)

Reflectance and Transmittance



Reflectance=
reflected power/incident power

$$R \equiv \frac{I_r A\cos\theta_r}{I_i A\cos\theta_i} = \frac{I_r}{I_i} = \left(\frac{E_{0r}}{E_{0i}}\right)^2 = r^2$$

Transmittance=
transmitted power/incident power

$$T \equiv \frac{I_t A\cos\theta_t}{I_i A\cos\theta_i} = \left(\frac{n_t \cos\theta_t}{n_i \cos\theta_i}\right)^2 t^2$$

$$R + T = 1$$

GaAs laser cleaved facet

(HW allows you to study the electromagnetic properties of the cavity without a gain medium.)

n=3.5

What does gain need to be for lasing?

Total internal reflection

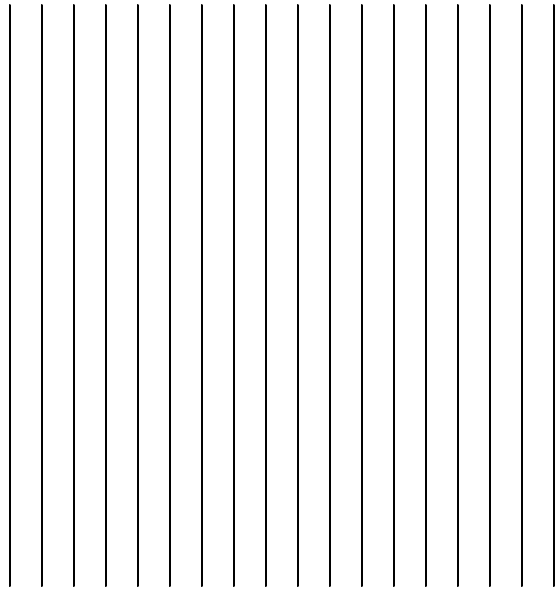
$$n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$$

Snell's law

(Discuss on board)

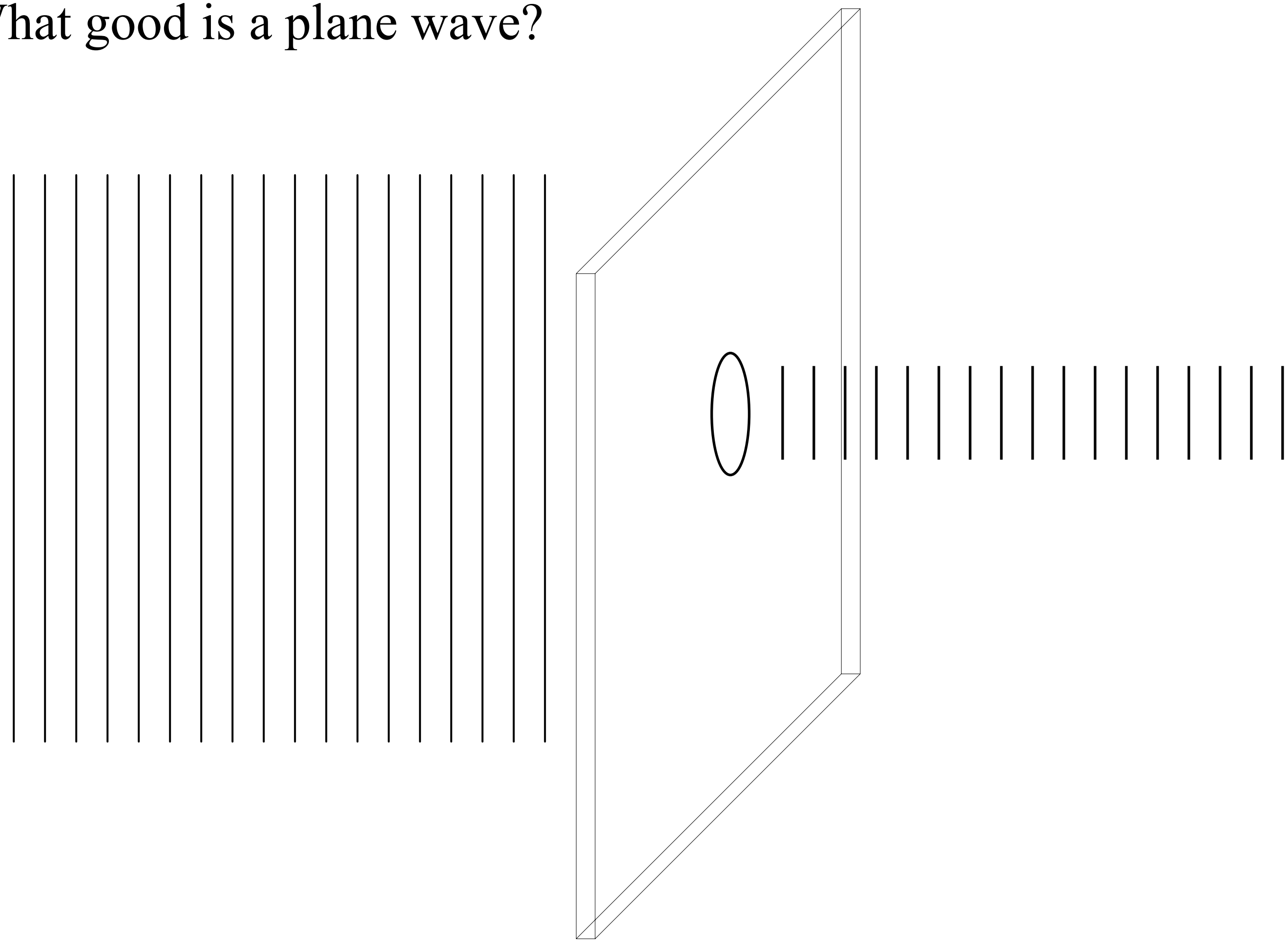
Ray tracing
(geometrical optics)

What good is a plane wave?



Ray tracing (geometrical optics)

What good is a plane wave?



Ray tracing (geometrical optics)

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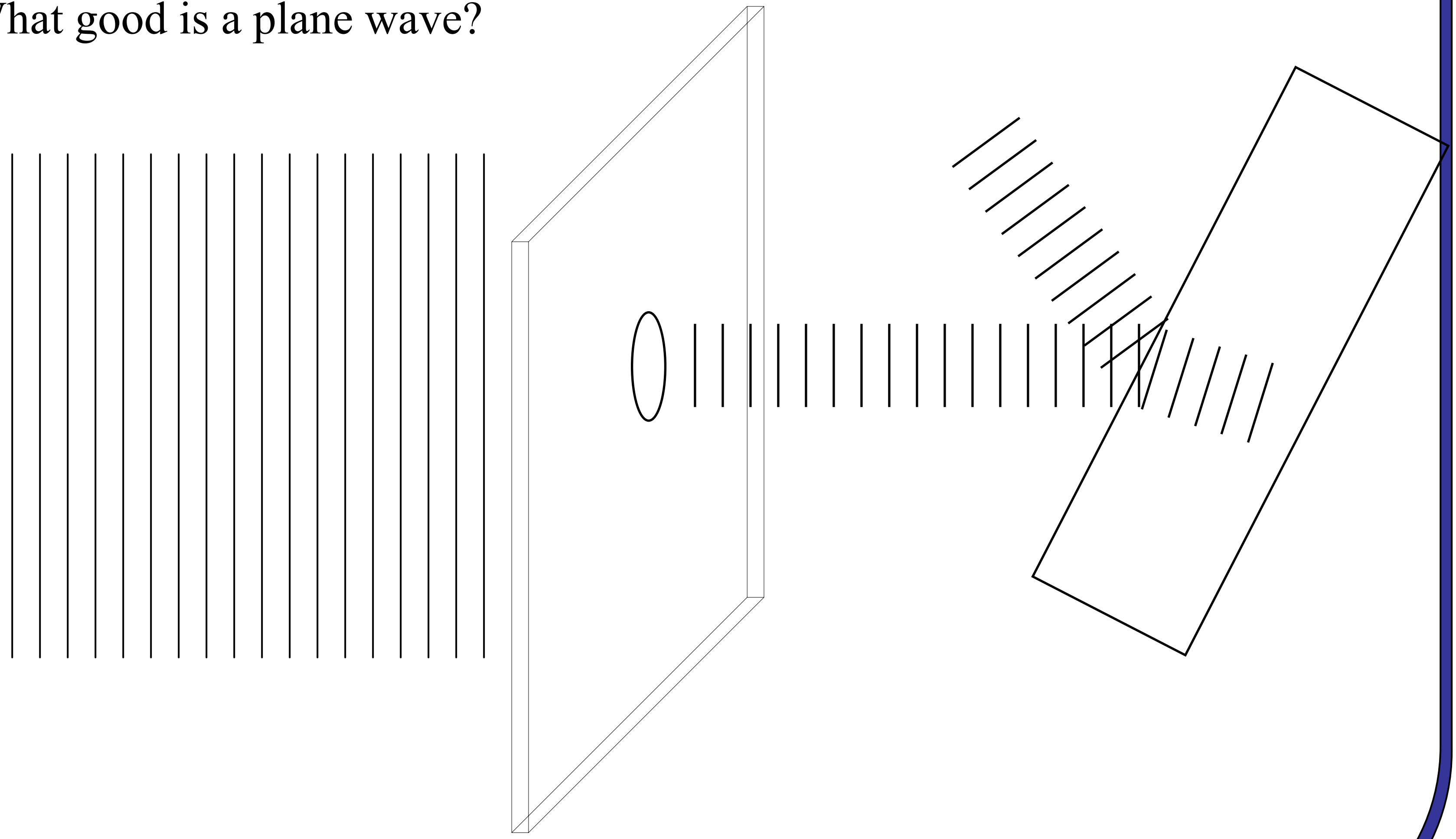
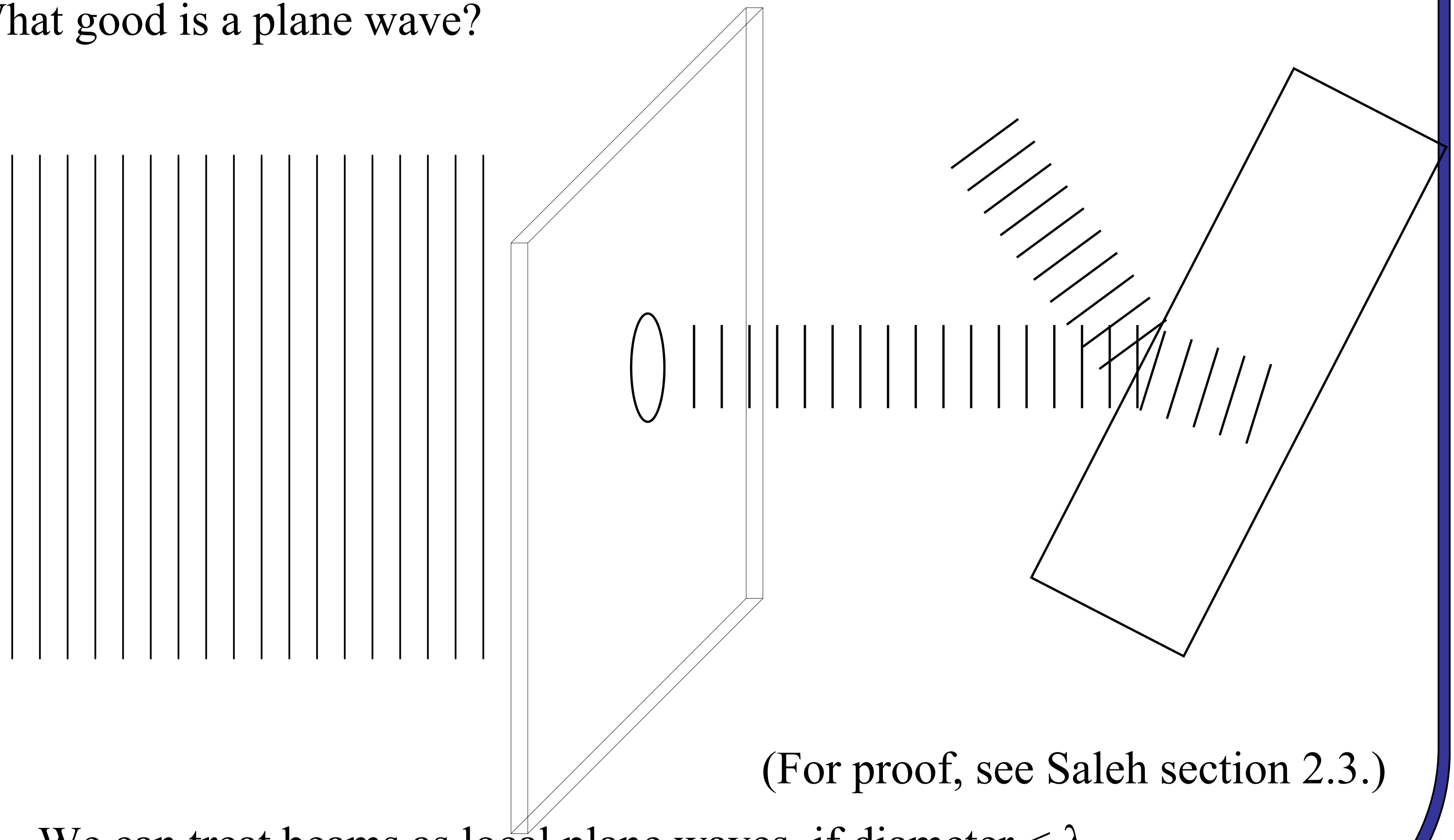


Ray tracing (geometrical optics)

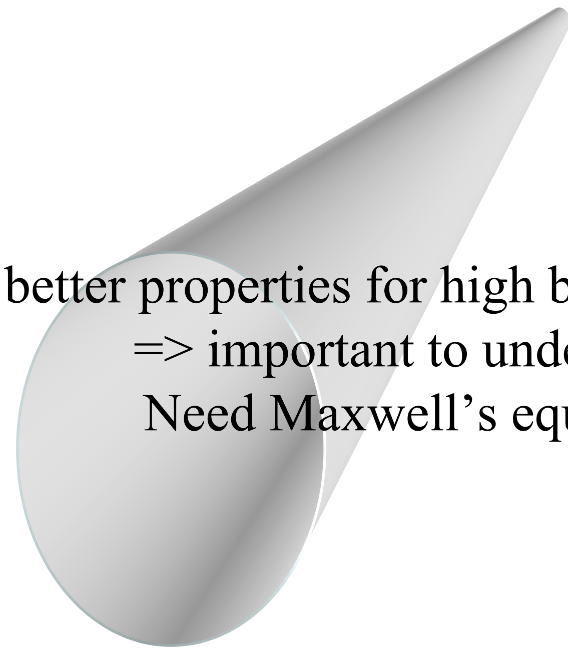
What good is a plane wave?



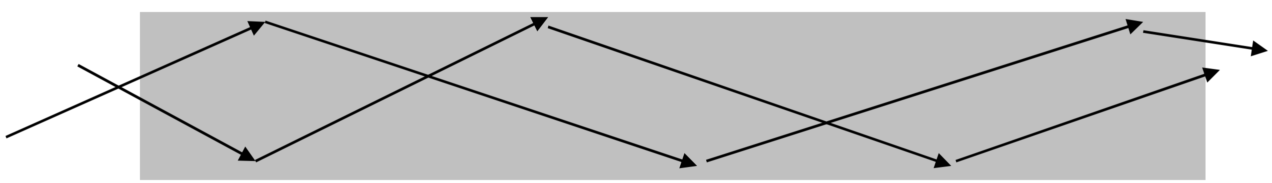
(For proof, see Saleh section 2.3.)

We can treat beams as local plane waves, if diameter < λ.
Otherwise, need to go back to Maxwell's equations! See lectures 5,6.

Example: optical fiber



Small diameter has better properties for high bit rate, long-haul communications
=> important to understand.
Need Maxwell's equations!



This description of fibers works only if diameter>λ.
Not always true!

Example: thick lens equation

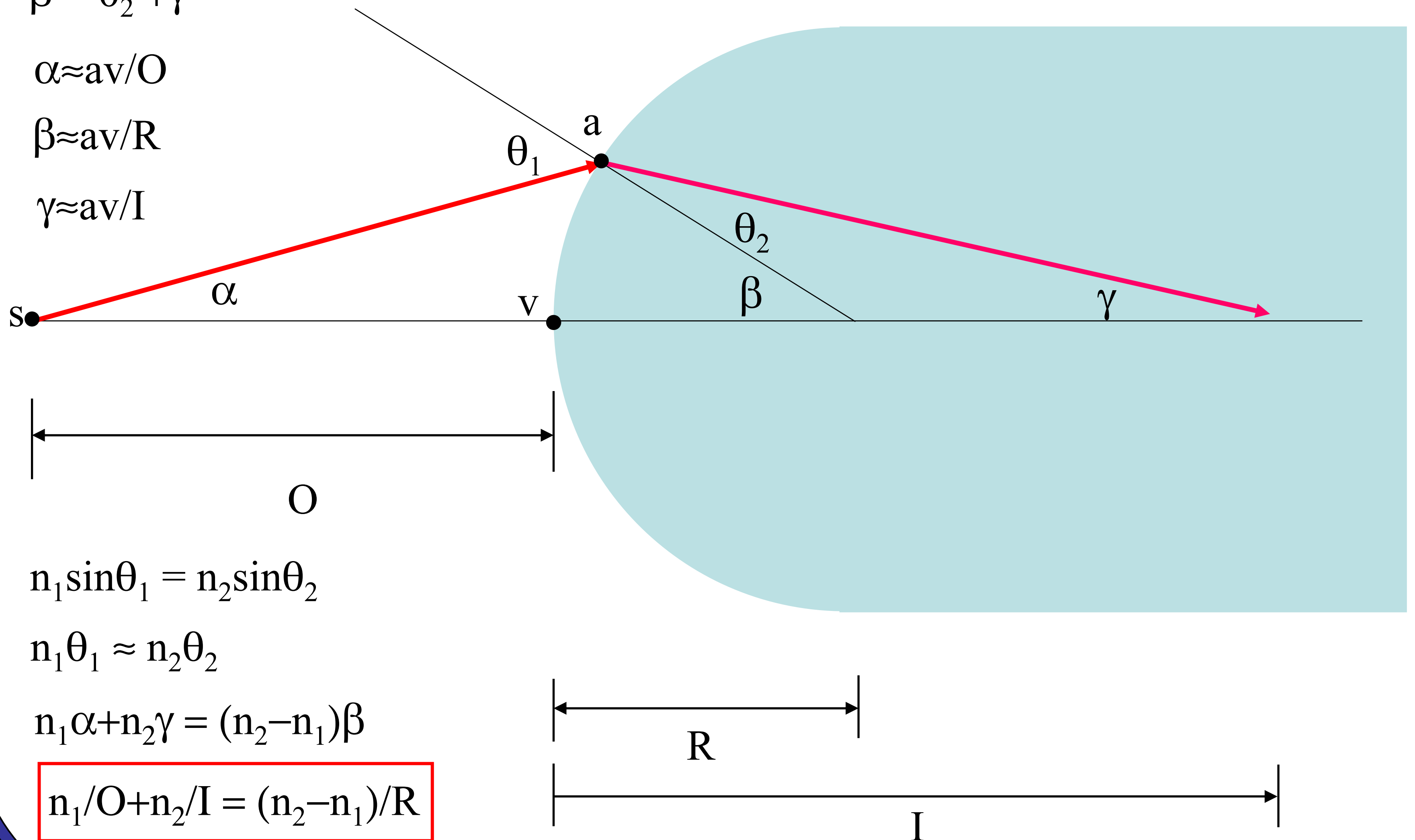
$\theta_1 = \alpha + \gamma$

$\beta = \theta_2 + \gamma$

$\alpha \approx av/O$

$\beta \approx av/R$

$\gamma \approx av/I$



$n_1 \sin\theta_1 = n_2 \sin\theta_2$

$n_1\theta_1 \approx n_2\theta_2$

$n_1\alpha + n_2\gamma = (n_2 - n_1)\beta$

$n_1/O + n_2/I = (n_2 - n_1)/R$

All of geometrical optics treated similarly.

Next week:

Photons

Atoms