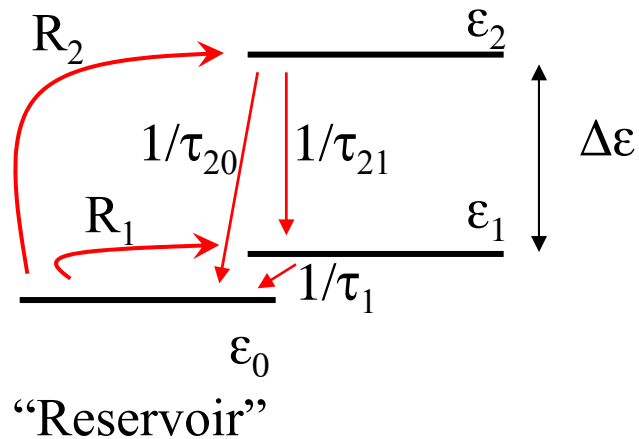


Lecture 6: Three-level rate equations

- Recall: No two-level laser (discuss)
- Three level laser: Let's try

(Note: Some audio missing because I went back in the presentation during the lecture.)

Three-level rate equations:



R_1 and R_2 are the external “pump” rates.

$1/\tau_{20}$ is spontaneous emission rate $2 \rightarrow 0$

$1/\tau_{21}$ is spontaneous emission rate $2 \rightarrow 1$

$1/\tau_2 = 1/\tau_{21} + 1/\tau_{20}$ spont em rate $2 \rightarrow$ (anything)

$1/\tau_1$ is spontaneous emission rate $1 \rightarrow 0$

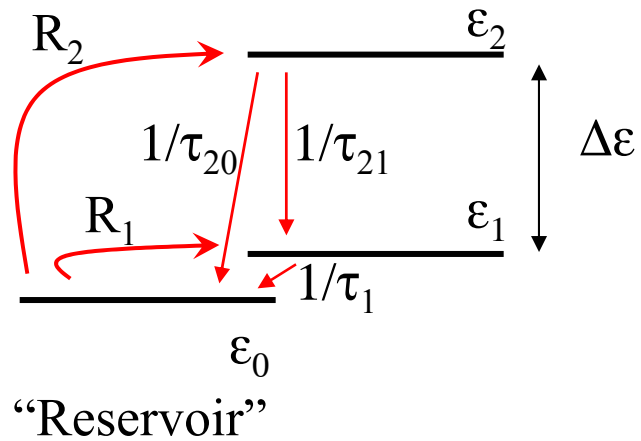
Goal: Use R_2 to achieve steady-state population inversion.

Guess:

Want lifetime in state 2 very long,

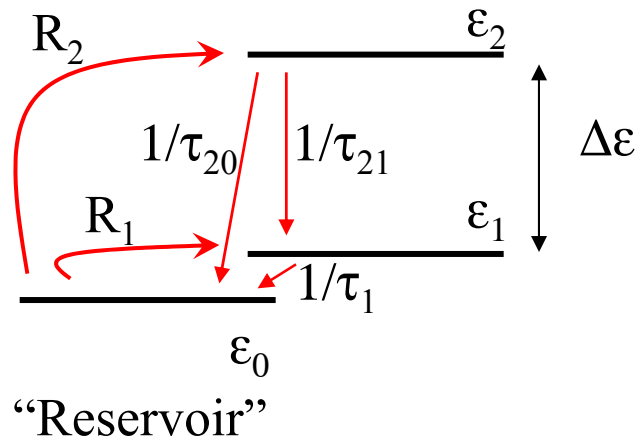
Want lifetime in state 1 very short.

Three-level rate equations:



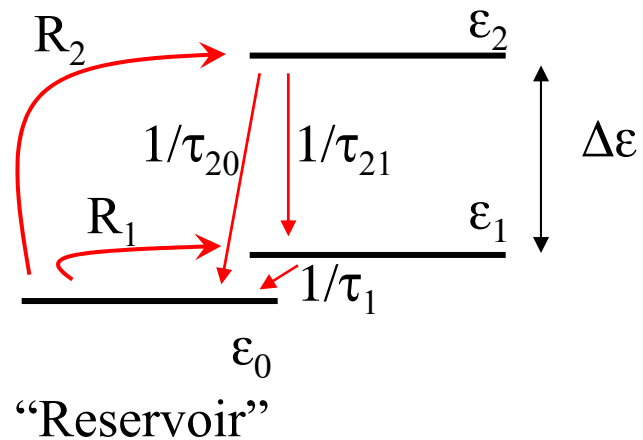
$$\frac{dN_2}{dt} = R_2(t)$$

Three-level rate equations:



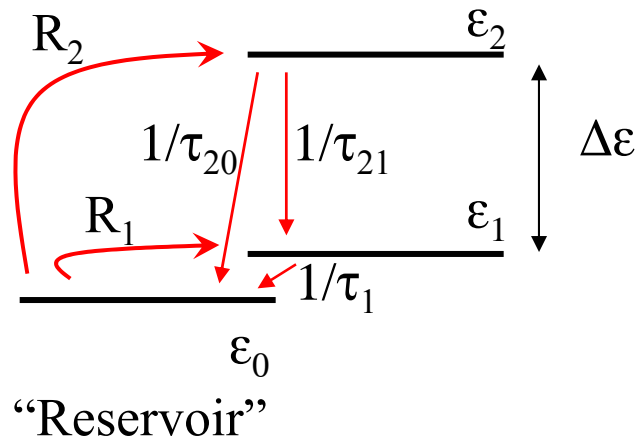
$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2} \right) N_2$$

Three-level rate equations:



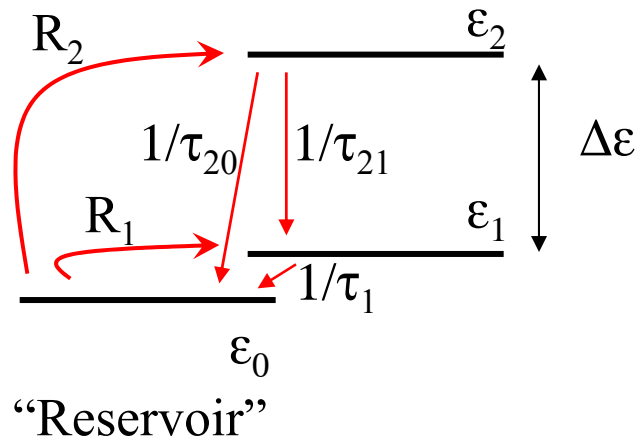
$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2} \right) N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$

Three-level rate equations:



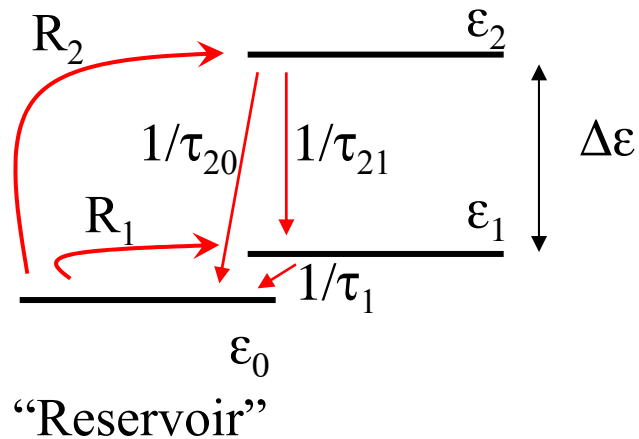
$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2} \right) N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$

Three-level rate equations:



$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

Three-level rate equations:



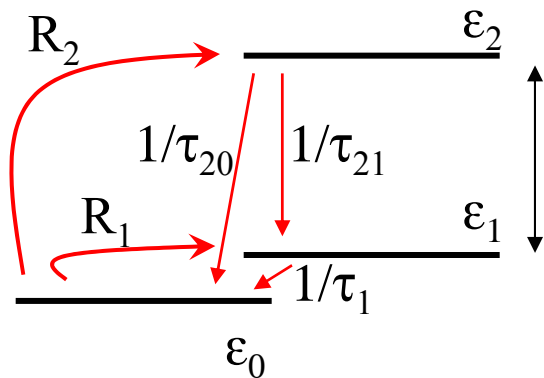
$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)]$$

$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

Four cases to consider:

- 1) $I = 0$
- 2) I finite, $\tau_1 = 0$
- 3) I is pulsed.
- 4) General conditions for steady state.

Case One: I=0

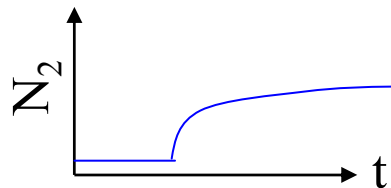
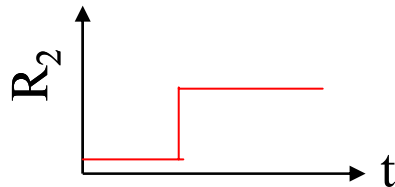


“Reservoir”

$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$

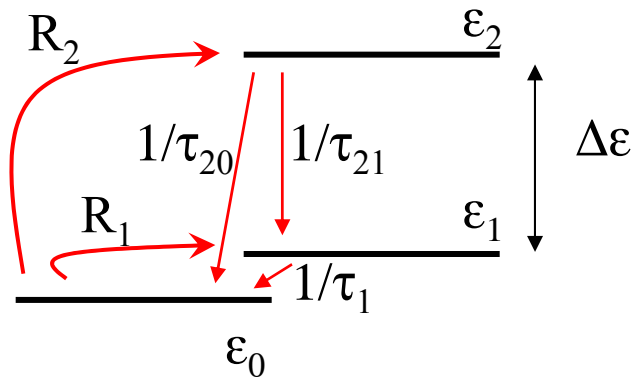
$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2$$

Let: $R_2(t) = R_{20}u(t)$ $N_2(0) = 0$



$$N_2(t) = \int \frac{dN_2}{dt} dt = R_{20} \tau_2 (1 - e^{-t/\tau_2})$$

Case One: I=0



“Reservoir”

$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu)(N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

0

0

$$\frac{dN_1}{dt} = +\left(\frac{1}{\tau_{21}}\right)N_2 - \left(\frac{1}{\tau_1}\right)N_1$$

$$\frac{dN_1}{dt} + \left(\frac{1}{\tau_1}\right)N_1 = +\left(\frac{1}{\tau_{21}}\right)N_2$$

$$\frac{dN_1}{dt} e^{t/\tau_1} + \left(\frac{1}{\tau_1}\right)N_1 e^{t/\tau_1} = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

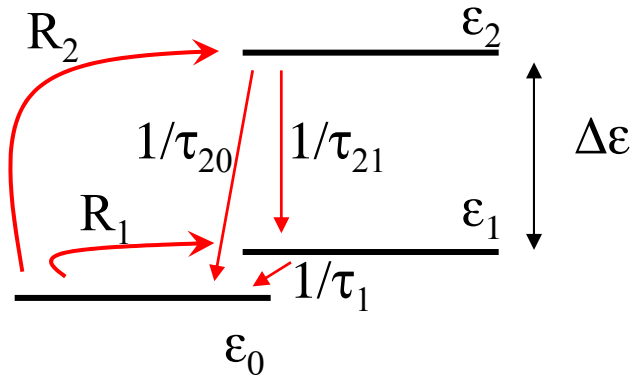
$$\frac{d}{dt}(N_1 e^{t/\tau_1}) = +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

$$\int \frac{d}{dt}(N_1 e^{t/\tau_1}) = \int +\left(\frac{1}{\tau_{21}}\right)N_2 e^{t/\tau_1}$$

$$N_1 e^{t/\tau_1} = \int +\left(\frac{1}{\tau_{21}}\right)R_{20}\tau_2(1 - e^{-t/\tau_2})e^{t/\tau_1}$$

Evaluating the integral on the r.h.s. gives...

Case One: I=0



“Reservoir”

Solution:

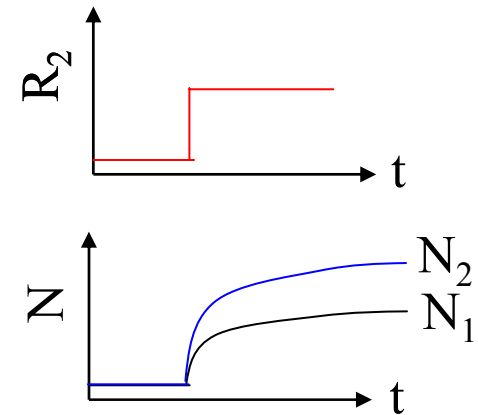
$$N_1(t) = \left(\frac{\tau_2}{\tau_{21}} \right) R_{20} \tau_1 \left\{ 1 + \frac{\tau_1 / \tau_2}{1 - \tau_1 / \tau_2} e^{-t/\tau_1} - \frac{1}{1 - \tau_1 / \tau_2} e^{-t/\tau_2} \right\}$$

$$N_2(t) = R_{20} \tau_2 \left(1 - e^{-t/\tau_2} \right)$$

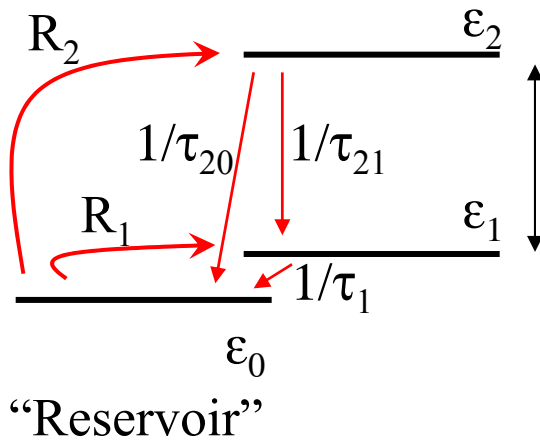
$$\tau_2 > \tau_1$$

needed for population inversion and gain!

Example: Nd:YAG 255 microseconds vs. 30 ns



Case Two: I is finite



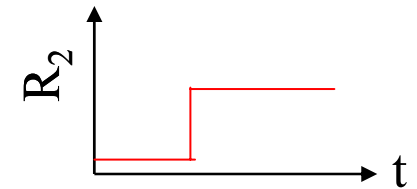
$$\Delta\varepsilon \frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_v}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)]$$

because $\tau_1=0$

$$\frac{dN_2}{dt} + \left(\frac{1}{\tau_2}\right)N_2 + \frac{I_v}{h\nu}[\sigma(\nu) \cdot (N_2)] = R_2(t)$$

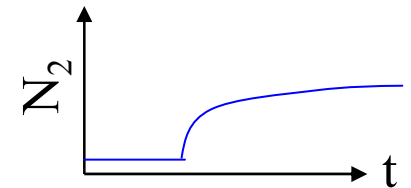
$$I_s \equiv \frac{h\nu}{\sigma\tau_2} \quad \text{“Saturation intensity”}$$

$$\frac{dN_2}{dt} + \frac{1}{\tau_2} \left[1 + \frac{I_v}{I_s}\right] N_2 = R_2(t) = R_{20}u(t)$$

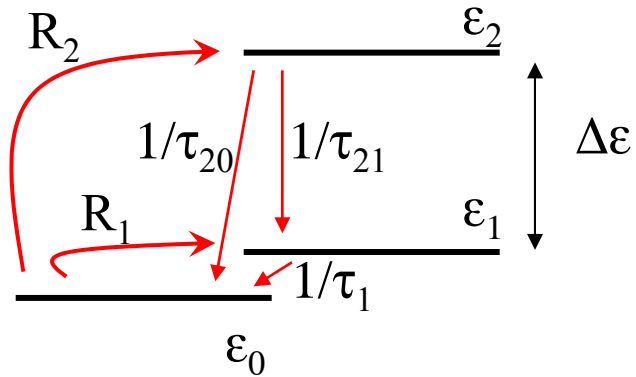


Solve as in case 1:

$$N_2(t) = \frac{R_{20}\tau_2}{1 + I_v/I_s} \left\{ 1 - e^{-\frac{t}{\tau_2}(1+I_v/I_s)} \right\}$$



Case One vs. Case Two



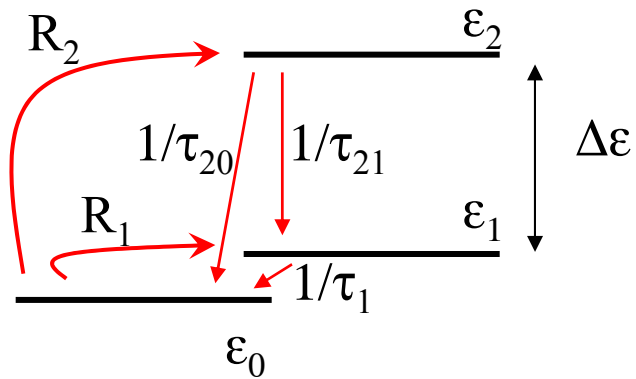
“Reservoir”

$$N_2(t) = R_{20} \tau_2 \left(1 - e^{-t/\tau_2} \right)$$

$$N_2(t) = \frac{R_{20} \tau_2}{1 + I_v / I_s} \left\{ 1 - e^{-\frac{t}{\tau_2} (1 + I_v / I_s)} \right\}$$

1. For $I_v \ll I_s$, results are the same.
2. For $I_v = I_s$, results are different by exactly a factor of two.
3. For $I_v \gg I_s$, N_2 is much less than it would be if $I_v = 0$.

Case One vs. Case Two



“Reservoir”

$$N_2(t) = R_{20} \tau_2 \left(1 - e^{-t/\tau_2} \right)$$

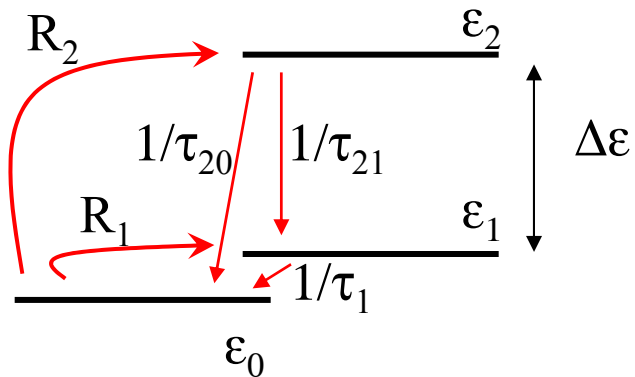
$$N_2(t) = \frac{R_{20} \tau_2}{1 + I_v / I_s} \left\{ 1 - e^{-\frac{t}{\tau_2} (1 + I_v / I_s)} \right\}$$

1. For $I_v \ll I_s$, results are the same.
2. For $I_v = I_s$, results are different by exactly a factor of two.
3. For $I_v \gg I_s$, N_2 is much less than it would be if $I_v = 0$.

This gives rise *gain saturation* in lasers.

Case Three: I an impulse

Assume $\tau_1 = 0$ again. So $N_1 = 0$.

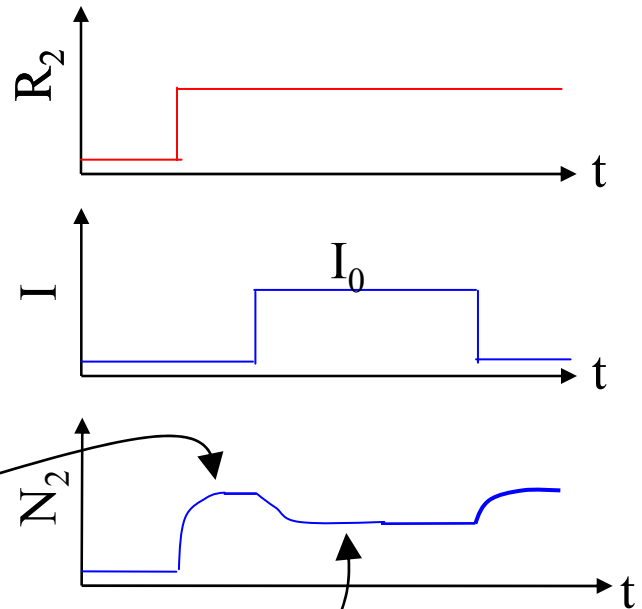


“Reservoir”

$$\frac{dN_2}{dt} + \frac{1}{\tau_2} \left[1 + \frac{I_v}{I_s} \right] N_2 = R_{20} u(t)$$

From case 1:

$$N_2(t) = R_{20} \tau_2 \left(1 - e^{-t/\tau_2} \right)$$

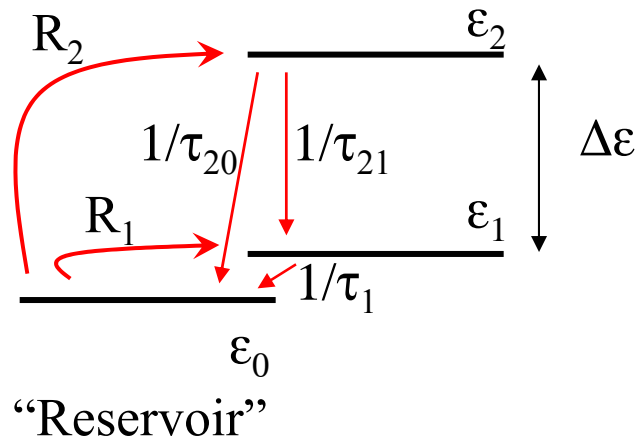


Now we have to solve equation on top again, with new initial condition.

$$N_2(t) = \frac{R_{20} \tau_2}{1 + I_v / I_s} \left\{ \left(\frac{I_v}{I_s} \right) e^{-\frac{t}{\tau'_2}} + 1 \right\} \rightarrow \frac{R_{20} \tau_2}{1 + I_v / I_s} \quad \tau'_2 = \frac{\tau_2}{1 + I_0 / I_s}$$

So increase I decreases N_2 hence gain.

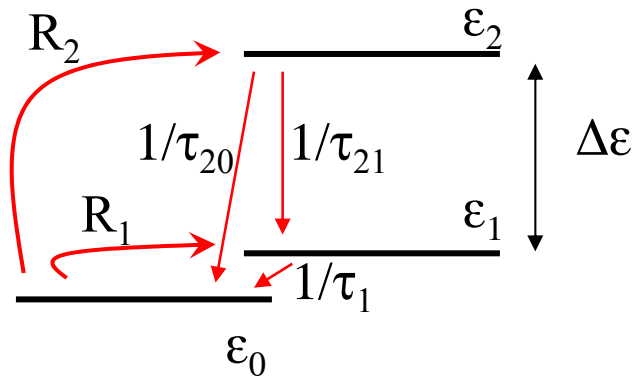
Steady state: $dN/dt=0$



$$\frac{dN_2}{dt} = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)]$$

$$\frac{dN_1}{dt} = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu} [\sigma(\nu) \cdot (N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

Steady state: $dN/dt=0$



“Reservoir”

Goal: Solve for N_1, N_2 :

$$\frac{dN_2}{dt} = 0 \Rightarrow$$

$$0 = R_2(t) - \left(\frac{1}{\tau_2}\right)N_2 - \frac{I_\nu}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)]$$

$$R_2(t) = \left(\frac{1}{\tau_2} + \frac{I_\nu \sigma(\nu)}{h\nu}\right)N_2 - \frac{I_\nu \sigma(\nu)}{h\nu}N_1$$

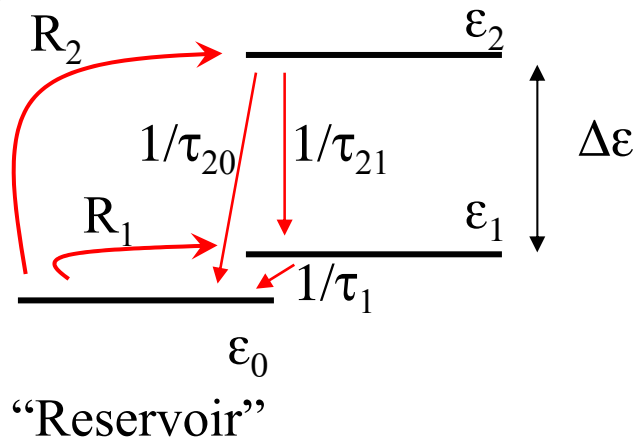
$$\frac{dN_1}{dt} = 0 \Rightarrow$$

$$0 = R_1(t) + \left(\frac{1}{\tau_{21}}\right)N_2 + \frac{I_\nu}{h\nu}[\sigma(\nu) \cdot (N_2 - N_1)] - \left(\frac{1}{\tau_1}\right)N_1$$

$$R_1(t) = -\left(\frac{1}{\tau_{21}} + \frac{I_\nu \sigma(\nu)}{h\nu}\right)N_2 + \left(\frac{I_\nu \sigma(\nu)}{h\nu} + \frac{1}{\tau_1}\right)N_1$$

Two equations, two unknowns.
Solve for N_1, N_2 .

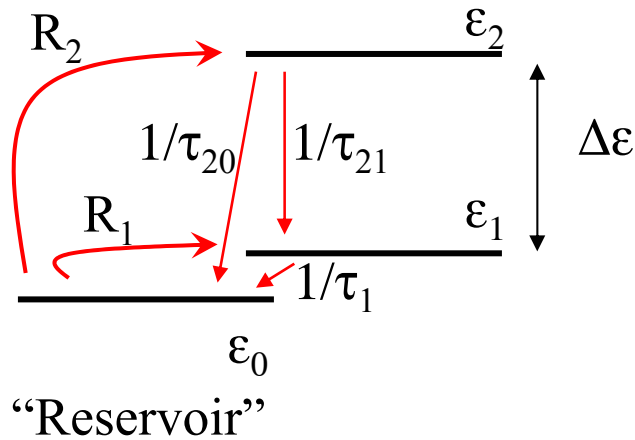
Steady state: $dN/dt=0$



$$N_1 = \frac{R_2 \left(\frac{1}{\tau_1} + \frac{I_v \sigma(\nu)}{h\nu} \right) + R_1 \left(\frac{I_v \sigma(\nu)}{h\nu} \right)}{\frac{1}{\tau_1 \tau_2} \left[1 + \left(\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{\tau_{21}} \right) \frac{I_v \sigma(\nu)}{h\nu} \right]}$$

$$N_2 = \frac{R_1 \left(\frac{1}{\tau_2} + \frac{I_v \sigma(\nu)}{h\nu} \right) + R_2 \left(\frac{1}{\tau_{21}} + \frac{I_v \sigma(\nu)}{h\nu} \right)}{\frac{1}{\tau_1 \tau_2} \left[1 + \left(\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{\tau_{21}} \right) \frac{I_v \sigma(\nu)}{h\nu} \right]}$$

Steady state: $dN/dt=0$



$$N_2 - N_1 = \frac{R_2 \tau_2 \left(1 - \frac{\tau_1}{\tau_{21}}\right) - R_1 \tau_1}{1 + \left(\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{\tau_{21}}\right) \frac{I_\nu \sigma(\nu)}{h\nu}}$$

$I=0$:

$$N_2 - N_1 = R_2 \tau_2 \left(1 - \frac{\tau_1}{\tau_{21}}\right) - R_1 \tau_1$$

$$\gamma = \sigma(\nu)(N_2 - N_1) \quad \gamma_0 = \sigma(\nu) R_2 \tau_2 \left(1 - \frac{\tau_1}{\tau_{21}}\right) - R_1 \tau_1$$

$$\gamma_0(I) = \frac{\gamma_0}{1 + I_\nu / I_s}$$

Saturation:

$$\frac{\partial I}{\partial z} = \gamma I_v \quad \gamma_0(I) = \frac{\gamma_0}{1 + I_v / I_s}$$

$$G \equiv \frac{I_2}{I_1}$$

$$\frac{\partial I}{\partial z} = \frac{\gamma_0}{1 + I_v / I_s} I_v$$

$$\frac{1 + I_v / I_s}{I_v} \partial I = \gamma_0 \partial z$$

$$\int_{I_1}^{I_2} \frac{1 + I_v / I_s}{I_v} \partial I = \int_{z=0}^{z=l_g} \gamma_0 \partial z$$

$$\ln \frac{I_2}{I_1} + \frac{I_2 - I_1}{I_s} = \gamma_0 l_g$$

$$\ln G + \frac{I_1}{I_s} (G - 1) = \gamma_0 l_g$$

Case 1: $I_1 \ll I_s$

$$\ln G = \gamma_0 l_g \Rightarrow G = e^{\gamma_0 l_g}$$

Case 2: $I_1 \gg I_s$

$$I_2 = I_1 + \gamma_0 l_g I_s$$

Conclusions for today:

- 2-level system: No population inversion steady state
- Now we know that 3-level system can have gain.
- Now we know how saturation of gain comes about.
- Next week, we'll consider laser oscillation.