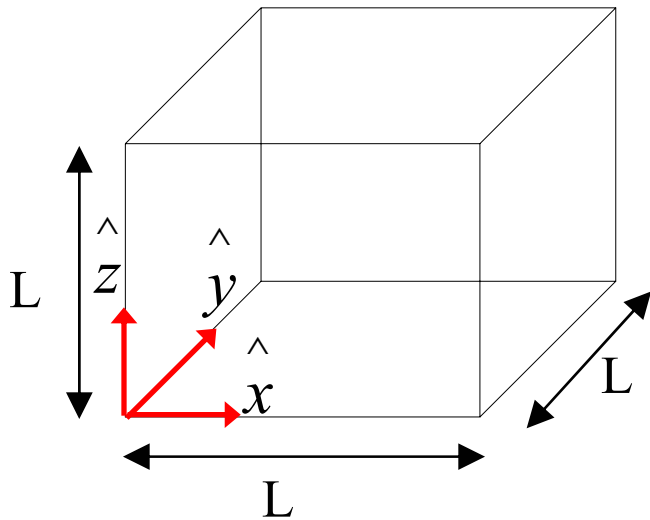


Lecture 4: Photons and atoms

- Electromagnetic modes in a box
- Blackbody radiation; *photons*, Planck law
- Photoelectric effect
- Energy spectrum of hydrogen
- Einstein A/B coefficients
- Three-level laser
- Reading: Ch. 7 of Verdeyen

Boundary conditions:



If we implement $\vec{\nabla} \cdot \vec{E} = 0$ we find two things:

First:

$$E_x(x, y, z, t) = E_1 \cos(k_x x) \cdot \sin(k_y y) \cdot \sin(k_z z) e^{i\omega t}$$

$$E_y(x, y, z, t) = E_2 \sin(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) e^{i\omega t}$$

$$E_z(x, y, z, t) = E_3 \sin(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) e^{i\omega t}$$

Second:

$$E_1 k_x + E_2 k_y + E_3 k_z = 0$$

$$\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k_z^2 = \left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2)$$

n_1, n_2, n_3 integers; at least two must be non-zero.

Each combination of n_1, n_2, n_3 is a “mode”.

Discuss superposition.

The magnitude of E_1, E_2, E_3 still can be any value!

(Subject to constraints above.)

How much energy is in the box?

Instantaneous energy per unit volume:

$$u(\vec{r}, t) = \frac{1}{2} \left(\epsilon_0 \left[\vec{E}(\vec{r}, t) \right]^2 + \mu_0 \left[\vec{H}(\vec{r}, t) \right]^2 \right)$$

Total energy in box:

$$U_{total} = \iiint_{box} u(\vec{r}, t) dV$$

It can be shown that for the box:

$$U_{total} = \frac{1}{2} (n_x^2 + n_y^2 + n_z^2) (E_1^2 + E_2^2 + E_3^2)$$

So, the amount of energy in the box can have any value.

We will show that this leads to a problem and must be wrong.

The energy in the box must be quantized: these are *photons*.

Concrete example of a mode:

$$n_x = 1; n_y = 1; n_z = 0 \Rightarrow k_x = \frac{\pi}{L}, k_y = \frac{\pi}{L}, k_z = 0$$

$$E_1 k_x + E_2 k_y + E_3 k_z = 0 \Rightarrow E_1 = E_2 = 0$$

$$E_x(x, y, z, t) = 0$$

$$E_y(x, y, z, t) = 0$$

$$E_z(x, y, z, t) = E_3 \sin\left(\frac{\pi x}{L}\right) \cdot \sin\left(\frac{\pi y}{L}\right) \cdot e^{i\omega t}$$

$$U_{total} = \frac{1}{2} (n_x^2 + n_y^2 + n_z^2) (E_1^2 + E_2^2 + E_3^2) = \frac{1}{2} E_3^2$$

The total energy in this mode can have a continuum of values, depending only on E_3 .

The same is true for all other modes.

(Discuss superposition).

Blackbody radiation

Consider E-M field in thermal equilibrium with matter at some temperature T .

If one is inside a box, do the walls glow?

Yes.

How is energy in the box
related to temperature?

According to the equipartition theorem from thermodynamics, every mode of the system has an average energy $\langle U \rangle = (1/2)k_B T$.

Note: This is already a problem. Energy infinite.

What is the energy per frequency, then we will integrate over frequencies?

There are many modes per unit frequency. Each has energy $k_B T$.

Modes per frequency

$$\varepsilon(\nu)d\nu = \frac{1}{2}kT \cdot N(\nu)d\nu$$

- $\varepsilon(\nu)d\nu$ is the energy between ν and $\nu+d\nu$.
- (This is the *spectrum* of the blackbody radiation.)
- $N(\nu)d\nu$ is the number of modes between ν and $\nu+d\nu$.

Modes per frequency $N(\nu)d\nu$

$$N(\nu)d\nu = \frac{1}{c^3}(8\pi)\nu^2 L^3$$

$$\varepsilon(\nu)d\nu = kT \cdot N(\nu)d\nu$$

$$\Rightarrow \varepsilon(\nu)d\nu = \frac{8\pi}{c^3} \cdot kT \cdot \nu^2 \cdot L^3$$

Rayleigh-Jeans law.

Experiments confirm at low frequencies only.

$$\Rightarrow \int_0^{\infty} \varepsilon(\nu)d\nu = \infty$$

Equipartition:

Recall Boltzmann factor $P(\varepsilon)$:

“The probability for a physical system to be in a state with energy ε is proportional to $e^{-\varepsilon/k_B T}$.”

(This is fundamentally linked to the concept of temperature.
Take it as an absolute truth for the whole class.)

Equipartition:

Recall Boltzmann factor $P(\varepsilon)$:

“The probability for a physical system to be in a state with energy ε is proportional to $e^{-\varepsilon/k_B T}$.”

In order to get $p(\varepsilon)$ to be between 0, 1 we need to normalize it: $\sum_i p(\varepsilon_i) = 1$

Boltzmann distribution:

Recall Boltzmann factor $P(\varepsilon)$:

“The probability for a physical system to be in a state with energy ε is proportional to $e^{-\varepsilon/k_B T}$.”

In order to get $p(\varepsilon)$ to be between 0, 1 we need to normalize it: $\sum_i p(\varepsilon_i) = 1$

$$p(\varepsilon) = \frac{e^{-\varepsilon/k_B T}}{\sum_i e^{-\varepsilon/k_B T}}$$

Boltzmann:

$$\langle \varepsilon \rangle = \sum_i \varepsilon_i p(\varepsilon_i) = \frac{\sum_i \varepsilon_i e^{-\varepsilon_i/k_B T}}{\sum_i e^{-\varepsilon_i/k_B T}}$$

$$p(\varepsilon) = \frac{e^{-\varepsilon/k_B T}}{\sum_i e^{-\varepsilon/k_B T}}$$

Equipartition theorem:

$$\langle \varepsilon \rangle = \frac{1}{2} kT$$

Based on classical statistical mechanics. (I did not prove.)

Consider the example mode:

$$n_x = 1; n_y = 1; n_z = 0 \quad U_{total} = \frac{1}{2} E_3^2$$

$$\langle \varepsilon \rangle = \frac{\int \frac{1}{2} E_3^2 \cdot e^{-\frac{1}{2} E_3^2 / k_B T} \cdot dE_3}{\int e^{-\frac{1}{2} E_3^2 / k_B T} \cdot dE_3} = \frac{1}{2} kT$$

Planck:

$$U_{total} = \frac{1}{2} E_3^2$$

“What if....”

$$U_{total} = \frac{1}{2} E_3^2 = nh\nu$$

n an integer, h “Planck’s constant”

Planck:

$$U_{total} = \frac{1}{2} E_3^2$$

“What if....”

$$U_{total} = \frac{1}{2} E_3^2 = nh\nu$$

$$\langle \varepsilon \rangle = \frac{\int \frac{1}{2} E_3^2 \cdot e^{-\frac{1}{2} E_3^2 / k_B T} \cdot dE_3}{\int e^{-\frac{1}{2} E_3^2 / k_B T} \cdot dE_3} = \frac{1}{2} kT$$

Planck:

$$\langle \varepsilon \rangle = \sum_i \varepsilon_i p(\varepsilon_i) = \frac{\sum_i \varepsilon_i e^{-\varepsilon_i/k_B T}}{\sum_i e^{-\varepsilon_i/k_B T}} = \frac{\sum_{n=0}^{\infty} nh\nu \cdot e^{-nh\nu/k_B T}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_B T}}$$

In HW#2, you will prove:

$$\langle \varepsilon \rangle = \frac{h\nu}{e^{h\nu/k_B T} - 1}$$

Planck at low frequency:

$$\langle \varepsilon \rangle = \frac{h\nu}{e^{h\nu/k_B T} - 1}$$

$$h\nu \ll k_B T :$$

$$h\nu / k_B T \ll 1$$

$$e^x \approx 1 + x \text{ for } x \ll 1$$

$$\langle \varepsilon \rangle = \frac{h\nu}{e^{h\nu/k_B T} - 1} \approx \frac{h\nu}{1 + h\nu/k_B T - 1} = k_B T$$

Equipartition

Planck at high frequency:

$$\langle \varepsilon \rangle = \frac{h\nu}{e^{h\nu/k_B T} - 1}$$

$$h\nu \gg k_B T :$$

$$\langle \varepsilon \rangle = \frac{h\nu}{e^{h\nu/k_B T} - 1} \approx \frac{h\nu}{e^{h\nu/k_B T}}$$

Not Equipartition!

Planck:

The “What if...”
is right!!!

@#\$%^&*()_!@

$$U_{total} = \frac{1}{2} E_3^2$$

“What if...”

$$U_{total} = \frac{1}{2} E_3^2 = nh\nu$$

$$\langle \varepsilon \rangle = \frac{\int \frac{1}{2} E_3^2 \cdot e^{-\frac{1}{2} E_3^2 / k_B T} \cdot dE_3}{\int e^{-\frac{1}{2} E_3^2 / k_B T} \cdot dE_3} = \frac{1}{2} kT$$

$$\langle \varepsilon \rangle = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1}$$

Modes per frequency

$$\varepsilon(\nu)d\nu = \frac{1}{2}kT \cdot N(\nu)d\nu$$

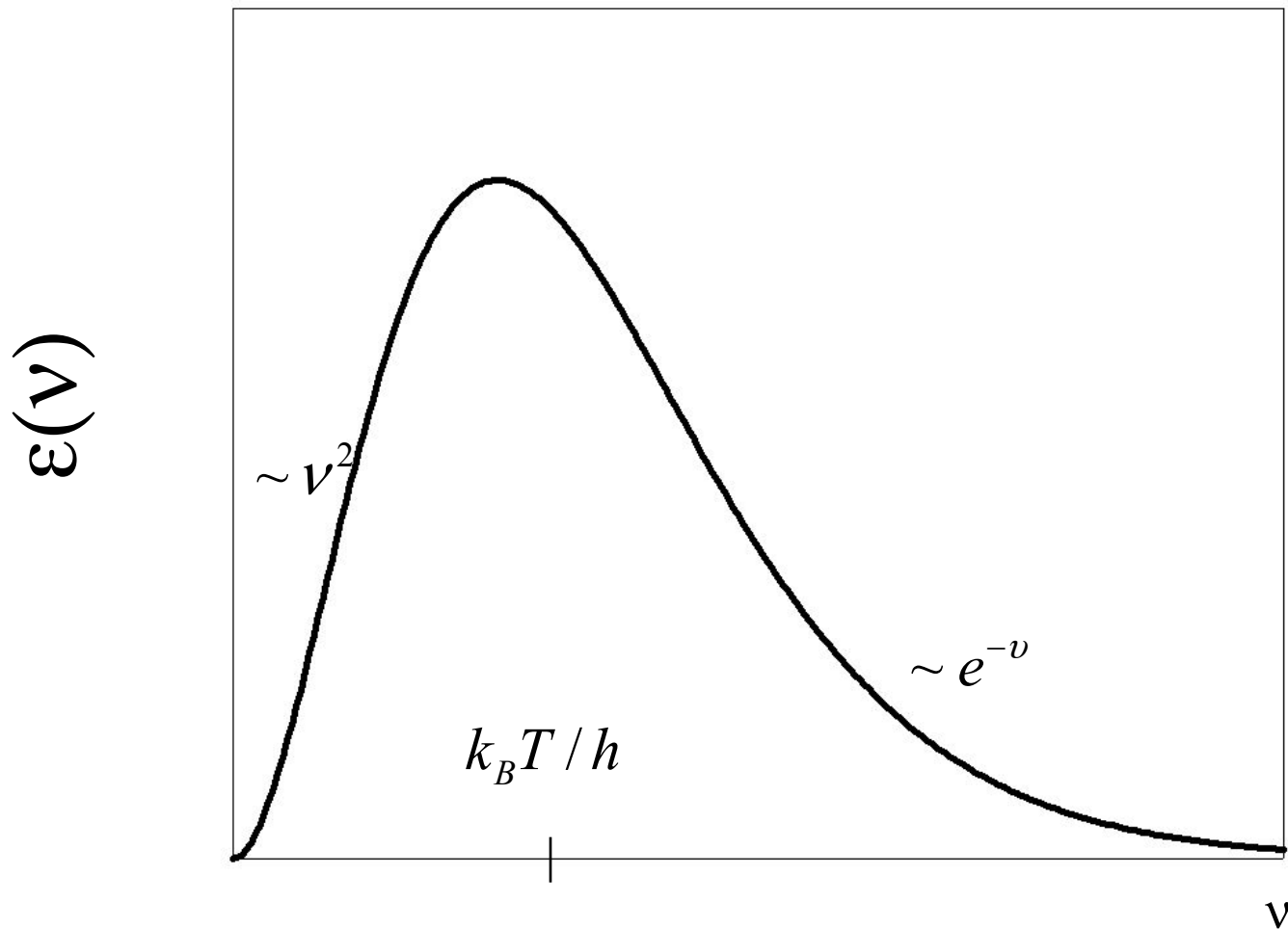
$$\varepsilon(\nu)d\nu = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1} \cdot N(\nu)d\nu$$

$$N(\nu)d\nu = \frac{1}{c^3} (8\pi)\nu^2 L^3 d\nu$$

$$\varepsilon(\nu)d\nu = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1} \cdot \frac{1}{c^3} (8\pi)\nu^2 L^3 d\nu$$

Note: my $\varepsilon(\nu)/L^3$ is Verdeyen's $\rho(\nu)$.

Planck spectrum:



You will calculate max. freq. in HW#2. Hint: it is in the infrared.

Intensity:

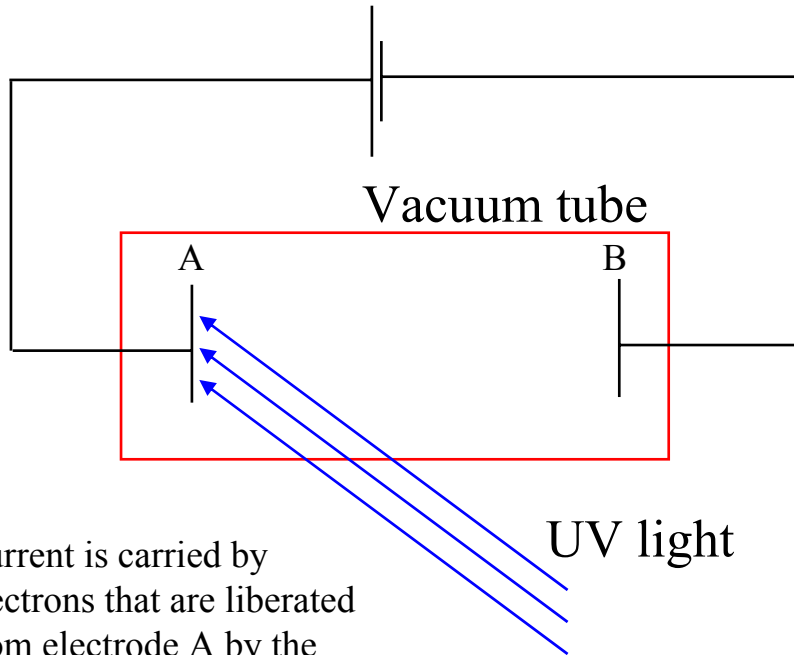
$$I(\nu)d\nu = (c/4) \cdot \varepsilon(\nu)d\nu / L^3 = \frac{h\nu}{e^{h\nu/k_B T} - 1} \cdot \frac{2\pi}{c^2} \nu^2 d\nu$$

=emitted power per unit area
in the frequency between ν
and $\nu+d\nu$

Conclusions:

- We have “discovered” photons.
- However, blacksmiths have known that hot metal glows red for hundreds of years.
- The arguments are for a “box” but the energy comes in quanta (photons) for any a.c. E-M field.
- Note the length of the box did not really matter.

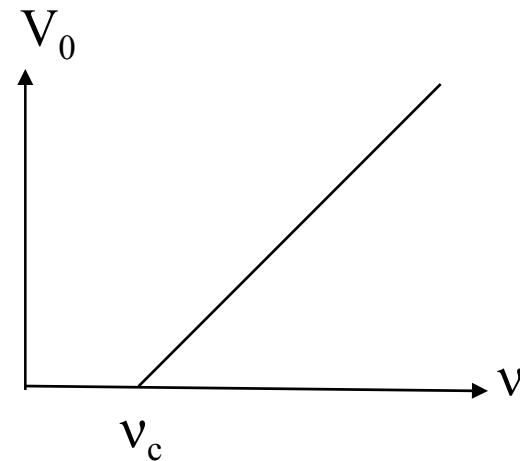
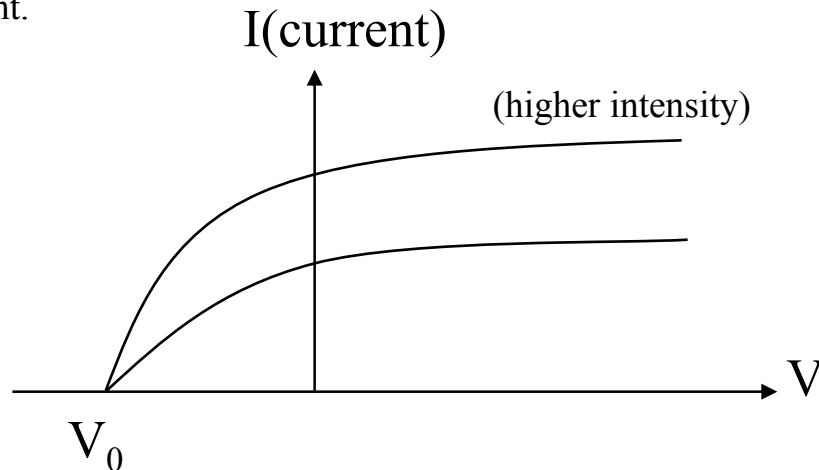
Photoelectric effect



Einstein's explanation:

- Electrons bound to metal by work function W
- If one photon is absorbed, energy of electron after being liberated is $h\nu - W = h(\nu - \nu_c)$
- eV_0 is the "stopping potential" = $h(\nu - \nu_c)$
- Slope of V_0 vs ν is h/e

Current is carried by electrons that are liberated from electrode A by the UV light.

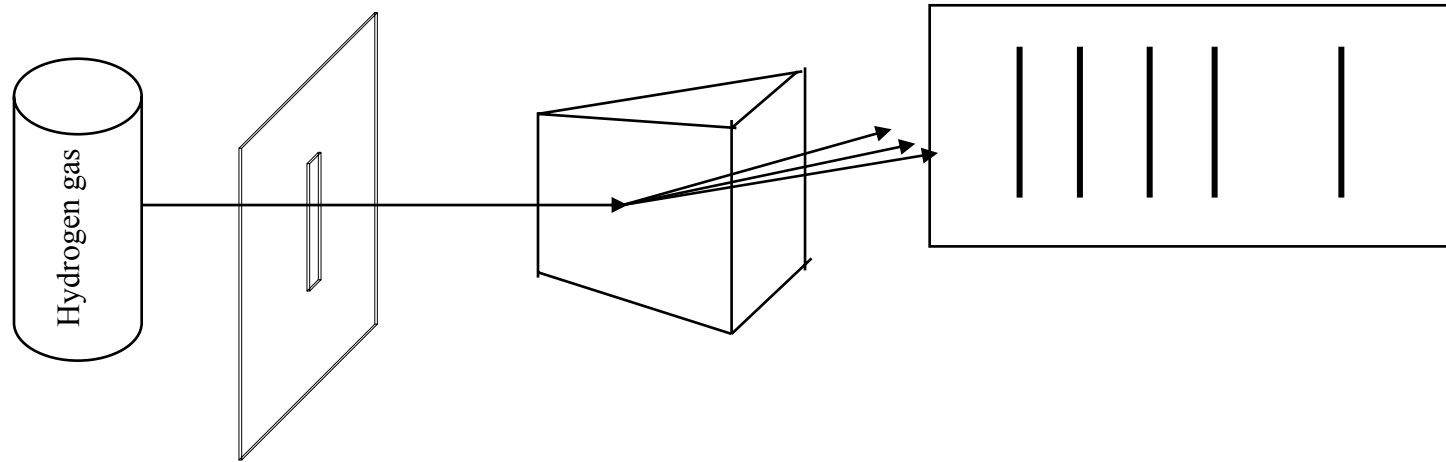


“Photons”

- Maxwell’s equations are still valid.
- However, the energy of any E/M wave is quantized:

$$\epsilon = n h \nu$$

Energy spectrum of hydrogen: Emission



Atoms excited by electrical discharge.

Lines are seen at well defined wavelengths.

For example:

$$\frac{1}{\lambda} = R \left(\frac{1}{4} - \frac{1}{n^2} \right) \quad n = 3, 4, 5, 6 \dots \quad R = 10^7 \text{ m}^{-1} \text{ (Rydberg)}$$

Others are seen:

$$\frac{1}{\lambda} = R \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \quad n > m$$

$m=1$ Lyman series; $m=2$ Balmer series, etc.

Energy spectrum of hydrogen: Absorption

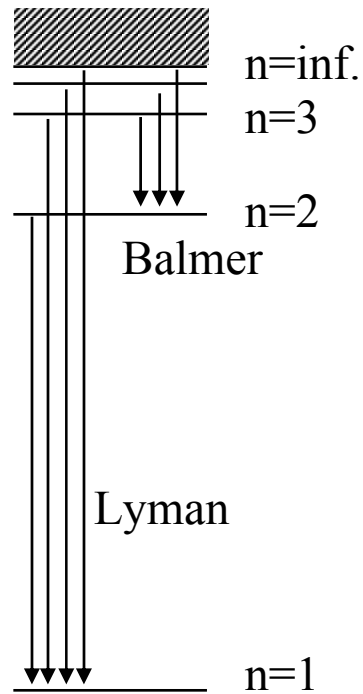
Certain wavelengths are strongly absorbed. These are the Lyman series and at elevated temperatures the Balmer series.

What does it mean? For photons, $\epsilon = n h \nu$

Hydrogen energy levels are *quantized*.

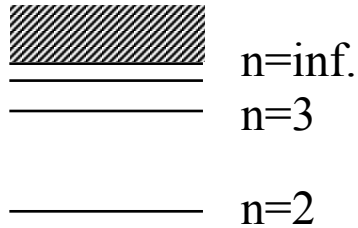
$$\Delta\epsilon_{n,m} = hcR \left(\frac{1}{m^2} - \frac{1}{n^2} \right) = \epsilon_{\text{final}} - \epsilon_{\text{initial}}$$

Energy spectrum of hydrogen:



$$\mathcal{E}_n = -13.6 \text{ eV} \cdot \frac{1}{n^2}$$

Energy spectrum of all atoms:



Similar to Hydrogen, but electron-electron interactions make them different.

Pauli exclusion principle:

No two electrons can occupy the same quantum state at the same time.

But the $n=1$ energy level has more than one quantum state.

$1s\uparrow$ $1s\downarrow$

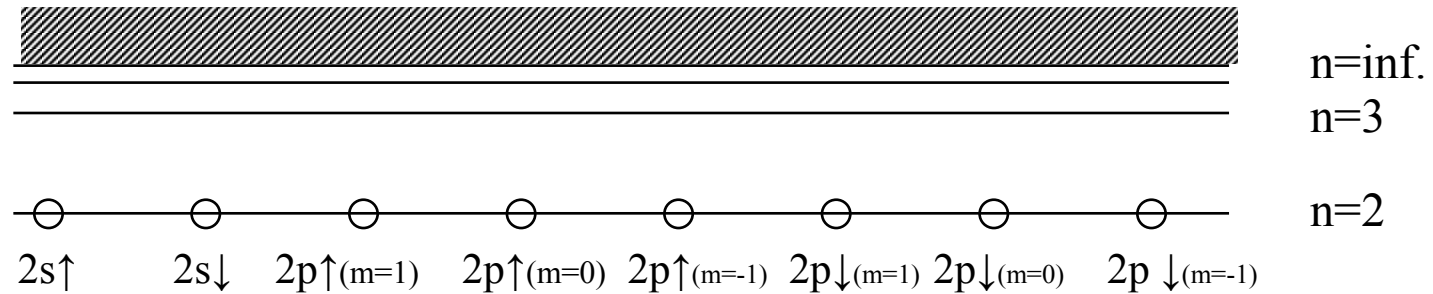
$n=2$ has more states:

$2s\uparrow$ $2s\downarrow$ $2p\uparrow(m=1)$ $2p\uparrow(m=0)$ $2p\uparrow(m=-1)$

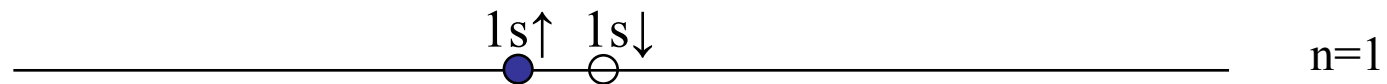
$2p\downarrow(m=1)$ $2p\downarrow(m=0)$ $2p\downarrow(m=-1)$

And so on...

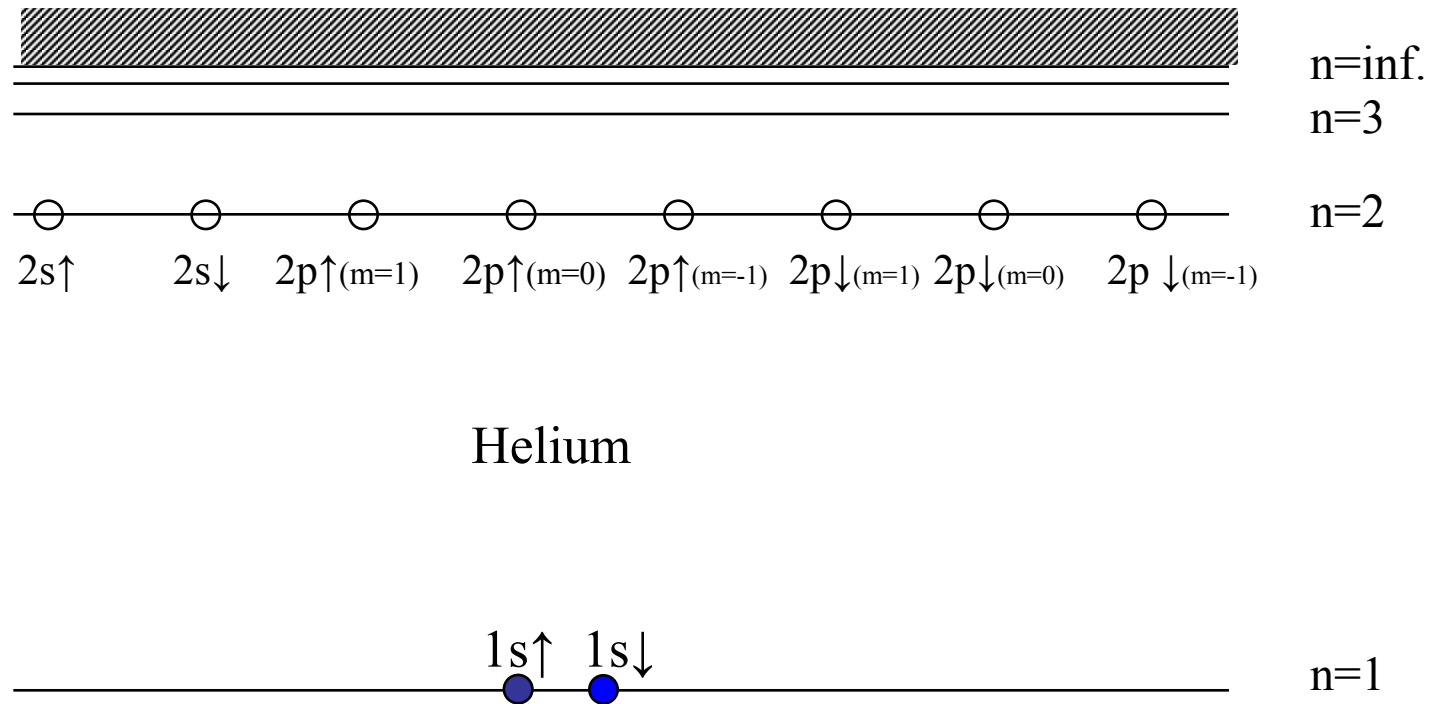
Energy spectrum of all atoms:



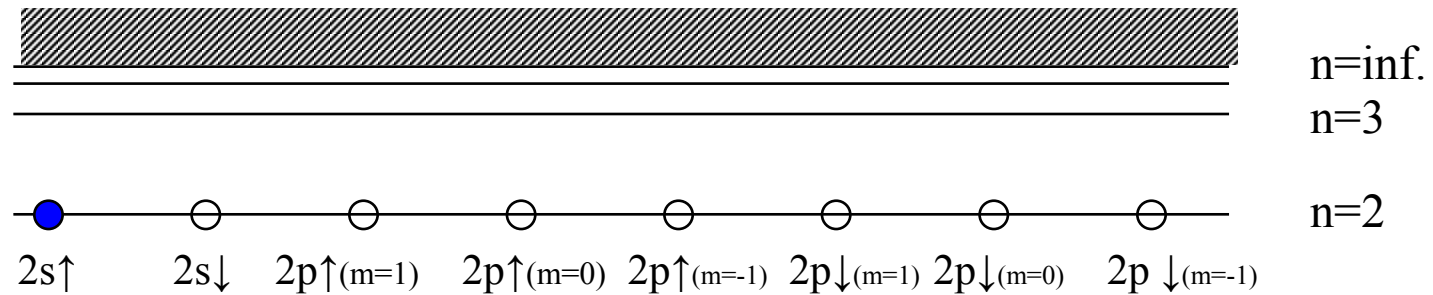
Hydrogen



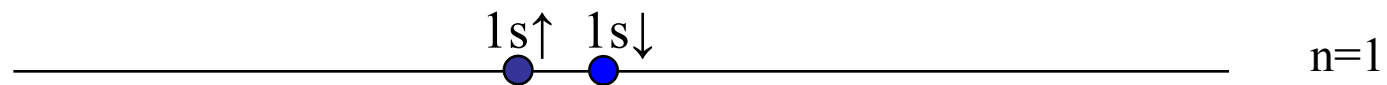
Energy spectrum of all atoms:



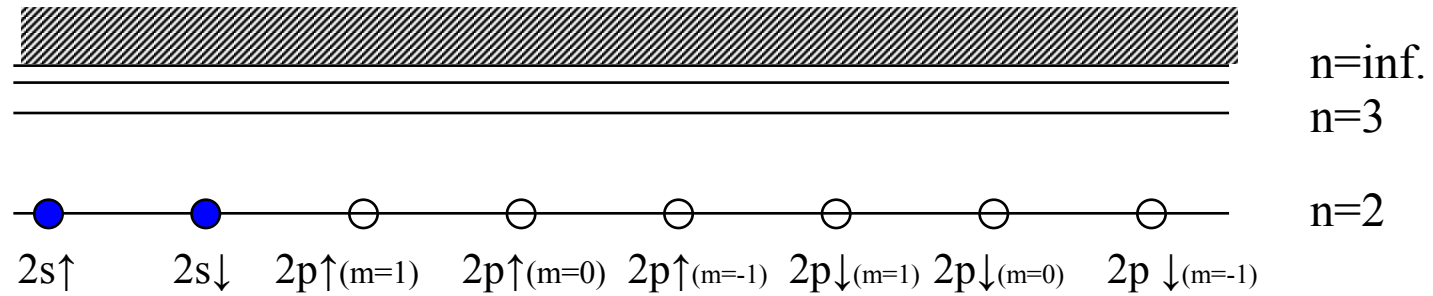
Energy spectrum of all atoms:



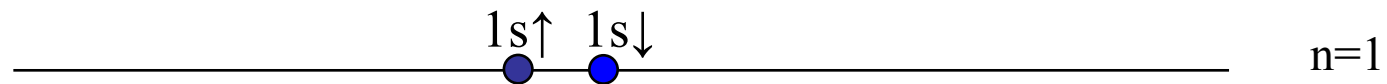
Lithium



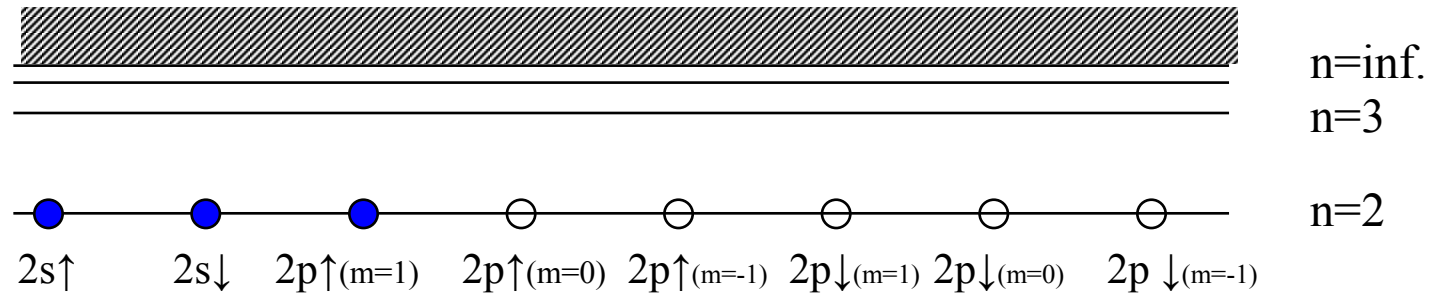
Energy spectrum of all atoms:



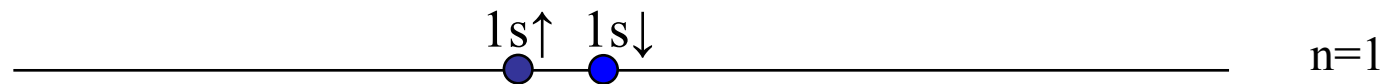
Beryllium



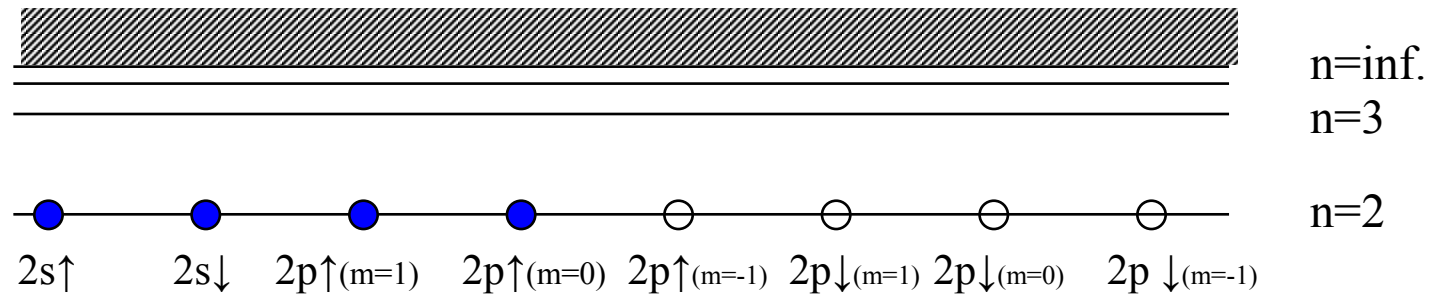
Energy spectrum of all atoms:



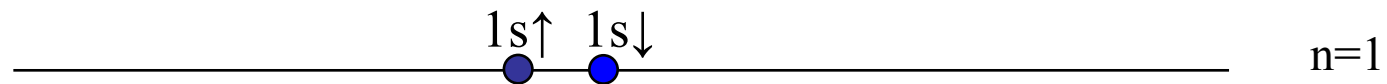
Boron



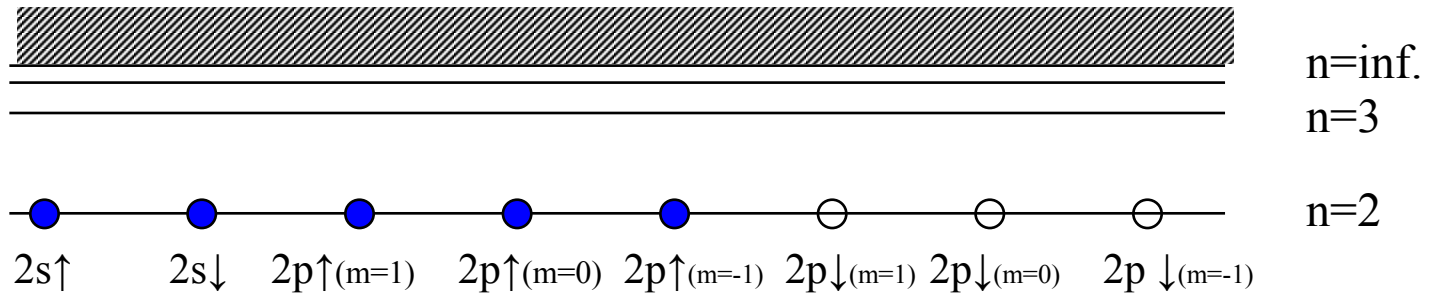
Energy spectrum of all atoms:



Carbon



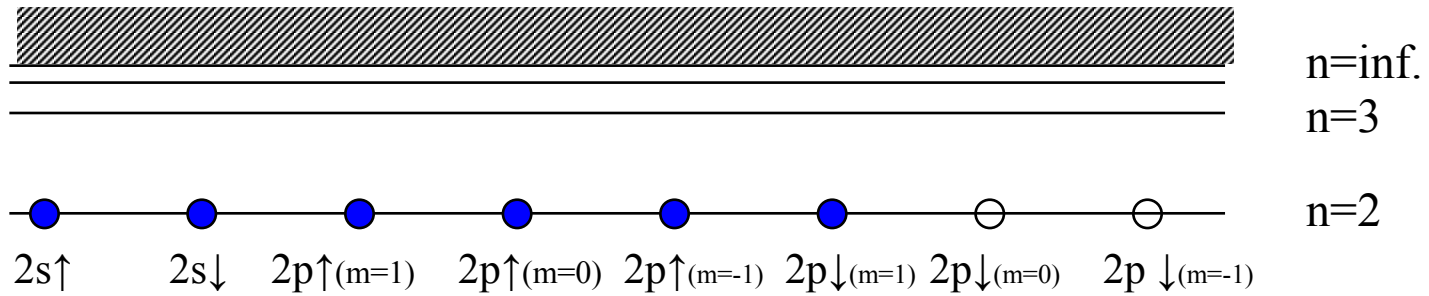
Energy spectrum of all atoms:



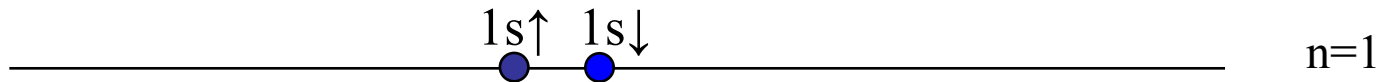
Nitrogen



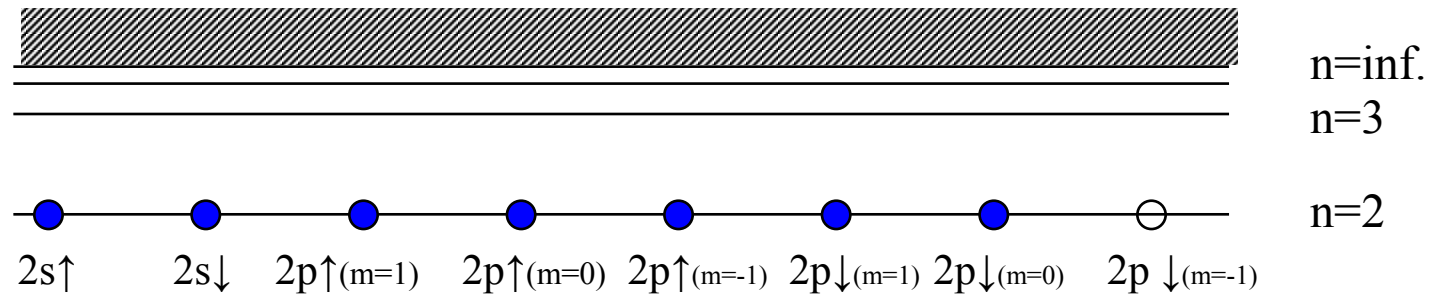
Energy spectrum of all atoms:



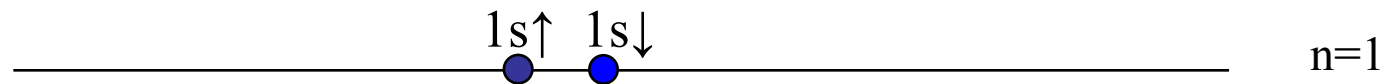
Oxygen



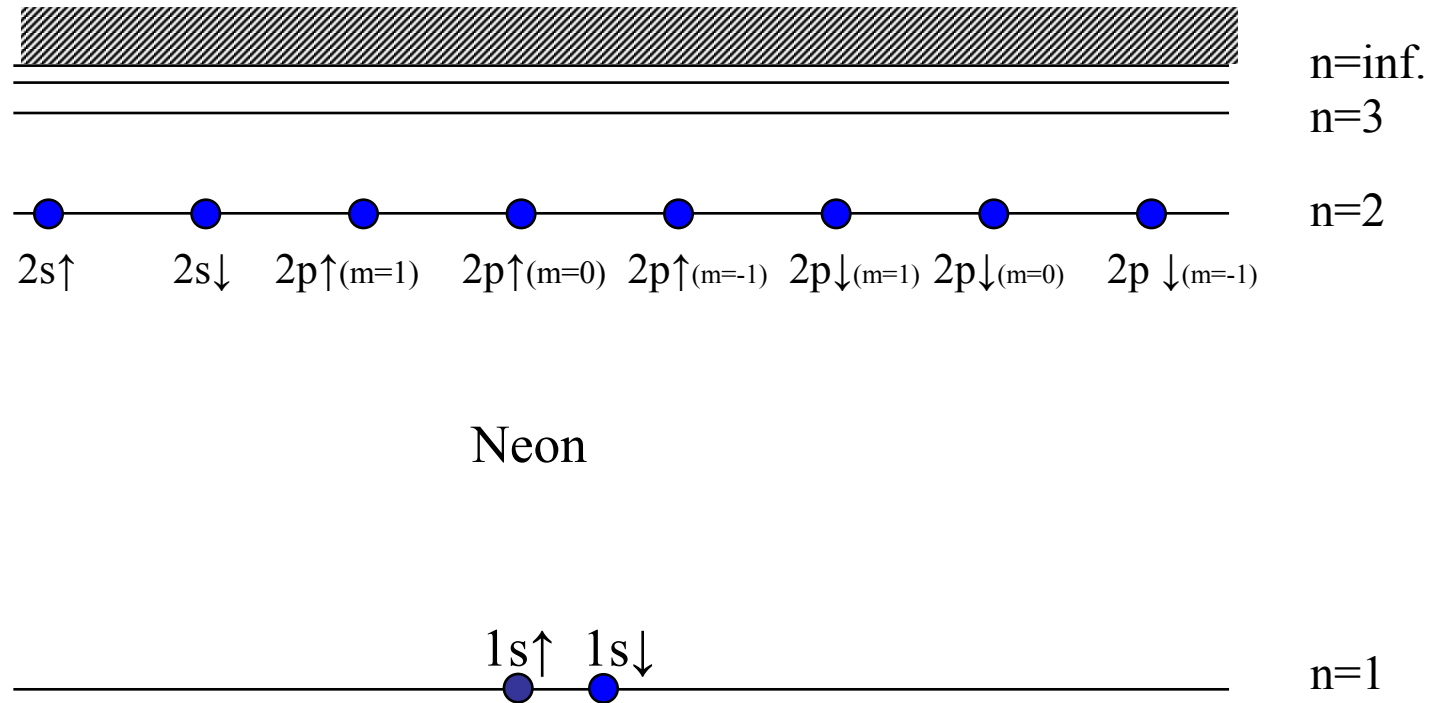
Energy spectrum of all atoms:



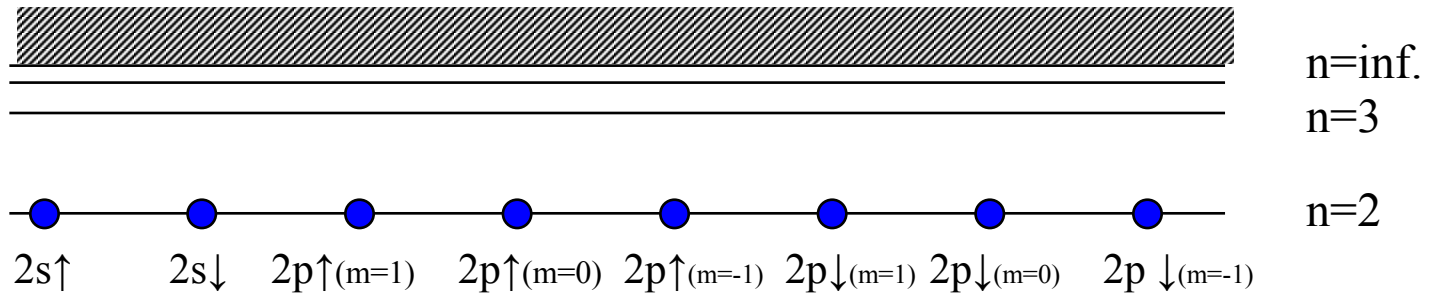
Flourine



Energy spectrum of all atoms:



Energy spectrum of all atoms:



And so on for all the elements...



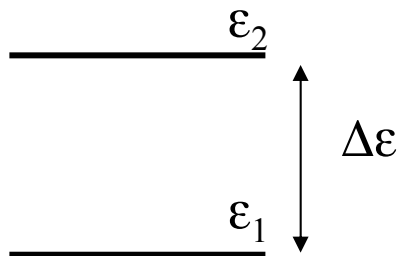
Important: In reality, all the $n=2$ states do not have the same energy!
Also, there are selection rules: only transitions between certain classes of states are allowed.

In a course on atomic physics, you would calculate and learn all the levels.

Einstein A/B coefficients

- Two-level atom (“two-level-onium”)
- Equilibrium occupation at temperature T
- Spontaneous emission rate
- Stimulated emission rate
- Absorption rate
- Relationship between all three rates

Two-level atom (“two-level-onium”)



Consider an ensemble of them.

Let N_1 be the # of atoms in state 1.

Let N_2 be the # of atoms in state 2.

Recall Boltzmann factor:

“The probability for a physical system to be in a state with energy ϵ is proportional to $e^{-\epsilon/k_B T}$.”

$$\Rightarrow N_2 / N_1 = e^{-\Delta\epsilon/k_B T}$$

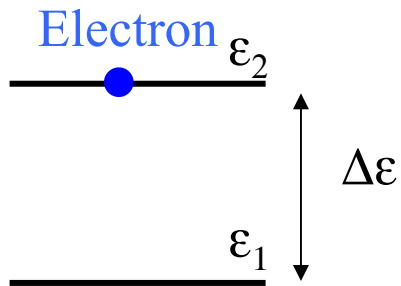
In thermal equilibrium, always more in state 1 than state 2.

This will mean later that we can't make a laser from a system in thermal equilibrium. Need a pump.

$$N_2 + N_1 = N_{Total} \Rightarrow \frac{N_2}{N_{Total}} = \frac{e^{-\Delta\epsilon/k_B T}}{1 + e^{-\Delta\epsilon/k_B T}}; \frac{N_1}{N_{Total}} = \frac{1}{1 + e^{-\Delta\epsilon/k_B T}}$$

Spontaneous emission:

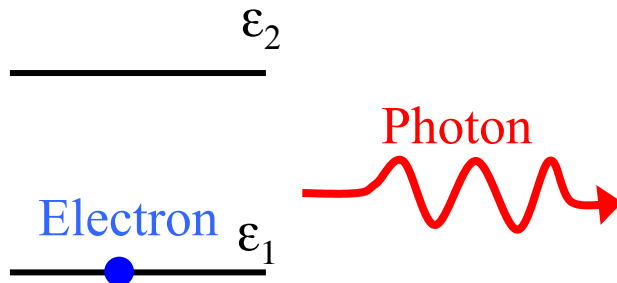
Before:



$$\frac{dN_2}{dt} = -AN_2$$

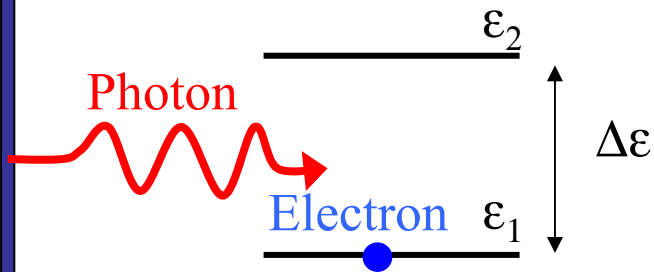
A is the decay rate.

After:



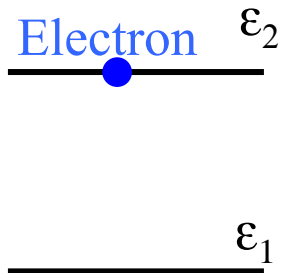
Absorption:

Before:



$$\frac{dN_2}{dt} = B_{12}N_1\rho(\nu)$$

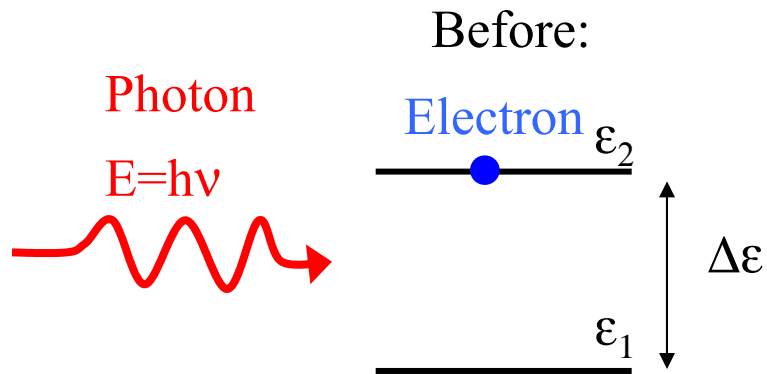
After:



B_{12} is a constant.

$\rho(\nu)$ is the E-M energy density.

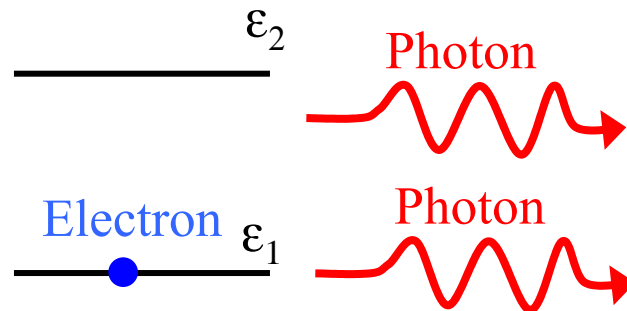
Stimulated emission:



$$\frac{dN_2}{dt} = -B_{21}N_2\rho(\nu)$$

B_{21} is a constant.

After:
First photon “stimulates”
emission of second photon



$\rho(\nu)$ is the E-M
energy density.

Add 'em up:

$$\frac{dN_2}{dt} = B_{12}N_1\rho(\nu) - B_{21}N_2\rho(\nu) - AN_2 = -\frac{dN_1}{dt}$$

Add 'em up:

$$\frac{dN_2}{dt} = B_{12}N_1\rho(\nu) - B_{21}N_2\rho(\nu) - AN_2 = -\frac{dN_1}{dt}$$

In thermal equilibrium, the average number N_2 and N_1 stay the same.

$$\frac{N_2}{N_{Total}} = \frac{e^{-\Delta\varepsilon/k_B T}}{1 + e^{-\Delta\varepsilon/k_B T}}; \frac{N_1}{N_{Total}} = \frac{1}{1 + e^{-\Delta\varepsilon/k_B T}}$$

Add 'em up:

$$\frac{dN_2}{dt} = B_{12}N_1\rho(\nu) - B_{21}N_2\rho(\nu) - AN_2 = -\frac{dN_1}{dt}$$

In thermal equilibrium, the average number N_2 and N_1 stay the same.

$$\frac{N_2}{N_{Total}} = \frac{e^{-\Delta\varepsilon/k_B T}}{1 + e^{-\Delta\varepsilon/k_B T}}; \frac{N_1}{N_{Total}} = \frac{1}{1 + e^{-\Delta\varepsilon/k_B T}}$$

In thermal equilibrium, we know $\rho(\nu)$ in a box!

$$\rho(\nu)d\nu = \frac{h\nu}{e^{h\nu/k_B T} - 1} \cdot \frac{1}{c^3} (8\pi)\nu^2 d\nu$$

Add 'em up:

$$\frac{dN_2}{dt} = B_{12}N_1\rho(\nu) - B_{21}N_2\rho(\nu) - AN_2 = -\frac{dN_1}{dt}$$

In thermal equilibrium, the average number N_2 and N_1 stay the same.

$$\frac{N_2}{N_{Total}} = \frac{e^{-\Delta\varepsilon/k_B T}}{1 + e^{-\Delta\varepsilon/k_B T}}; \frac{N_1}{N_{Total}} = \frac{1}{1 + e^{-\Delta\varepsilon/k_B T}}$$

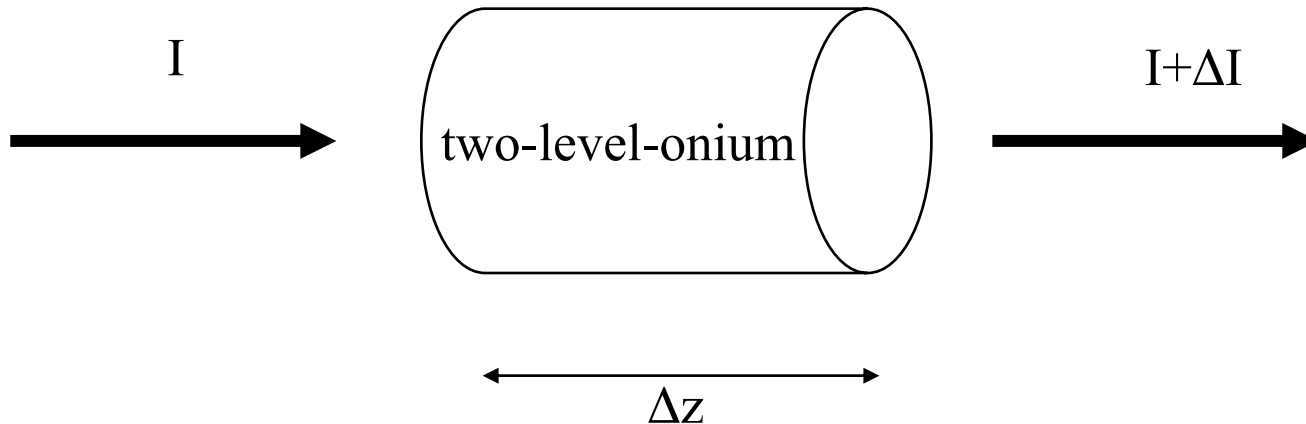
In thermal equilibrium, we know $\rho(\nu)$ in a box!

$$\rho(\nu)d\nu = \frac{h\nu}{e^{h\nu/k_B T} - 1} \cdot \frac{1}{c^3} (8\pi)\nu^2 d\nu$$

Einstein showed that this can only be true if:

$$B_{12} = B_{21} \quad \text{and} \quad A = \frac{8\pi h \nu^3}{c^3} B_{21}$$

Can we get gain?

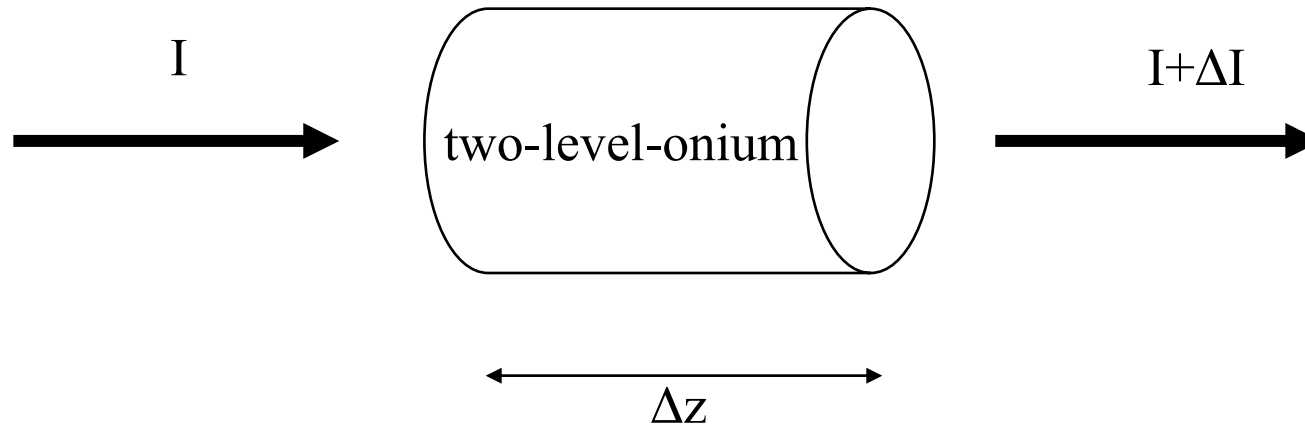


ΔI is due to spontaneous emission, stimulated emission, and absorption.

$$\Delta I_{stim.em.} = h\nu \cdot B_{21} N_2 \frac{I(\nu)}{c} \cdot \Delta z \quad \Delta I_{abs.} = -h\nu \cdot B_{12} N_1 \frac{I(\nu)}{c} \cdot \Delta z$$

$$\frac{\partial I}{\partial z} = \left[\frac{h\nu}{c} \cdot B_{21} (N_2 - N_1) \right] I(\nu)$$

Can we get gain?



$$\frac{\partial I}{\partial z} = \left[\frac{h\nu}{c} \cdot B_{21}(N_2 - N_1) \right] I(\nu)$$

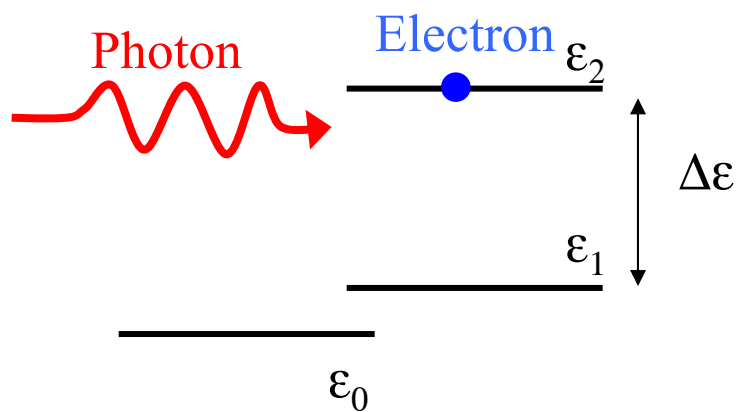
Need $N_2 > N_1$ for gain

If we put all atoms into excited state and pass the wave through a few times, they will eventually all end up half excited and half ground, and we will no longer have gain. (This can be shown rigorously.)

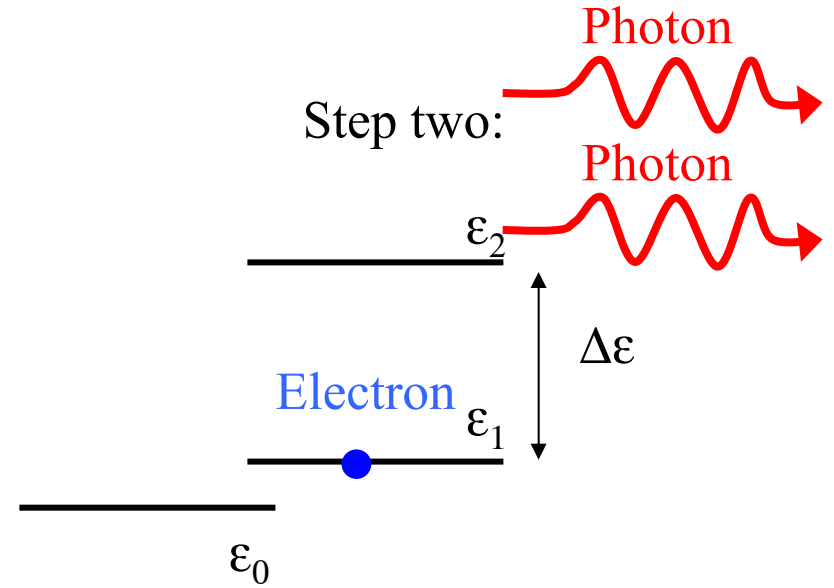
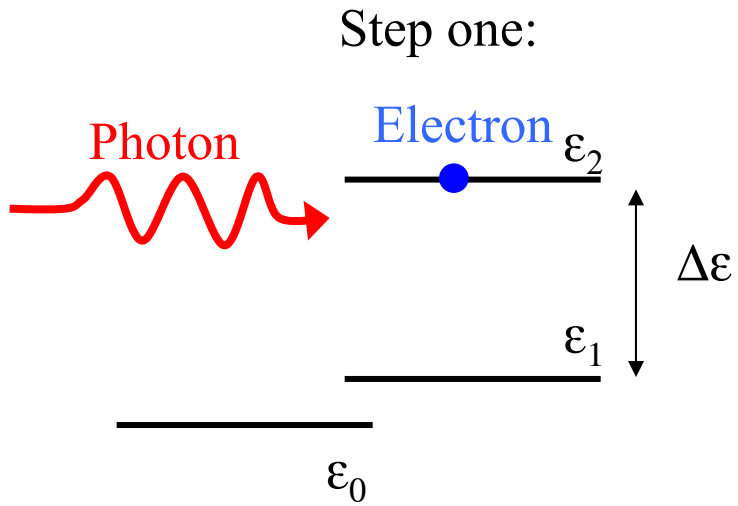
What is needed is a third level.

Three-level-onium:

Step one:

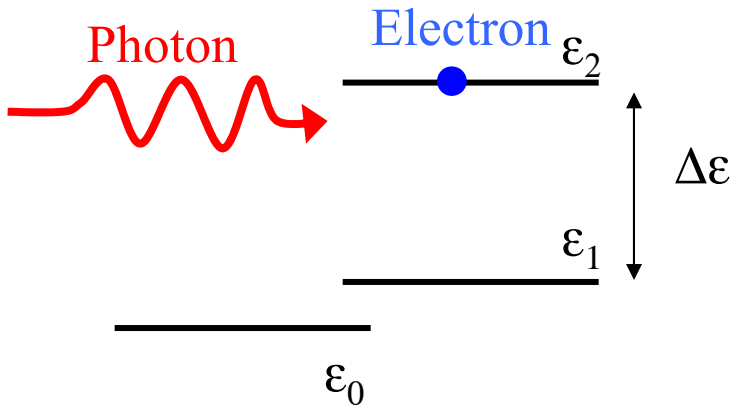


Three-level-onium:

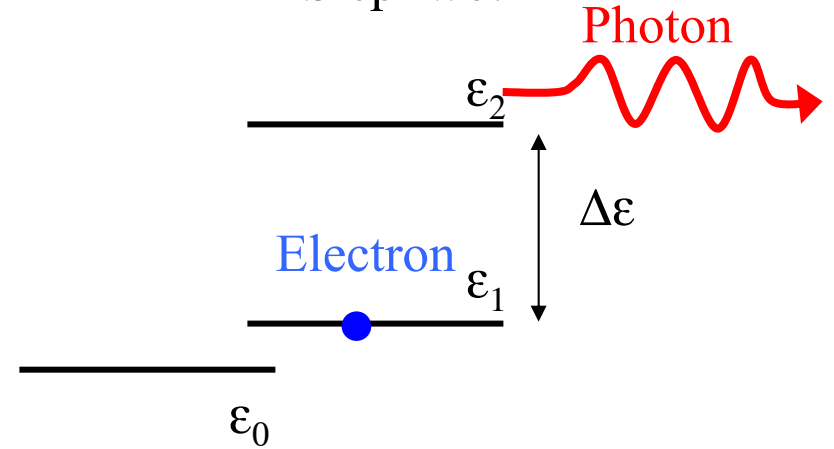


Three-level-onium:

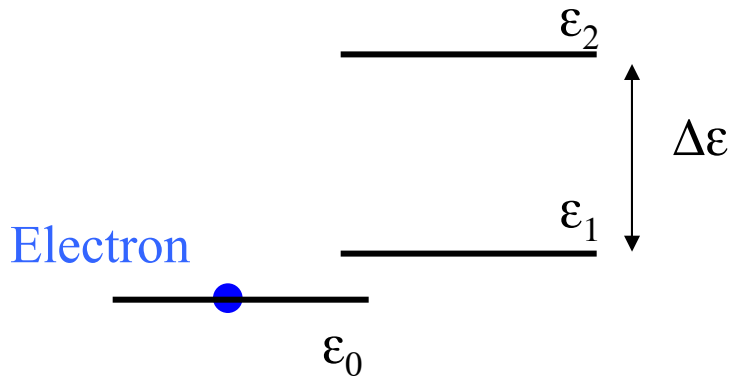
Step one:



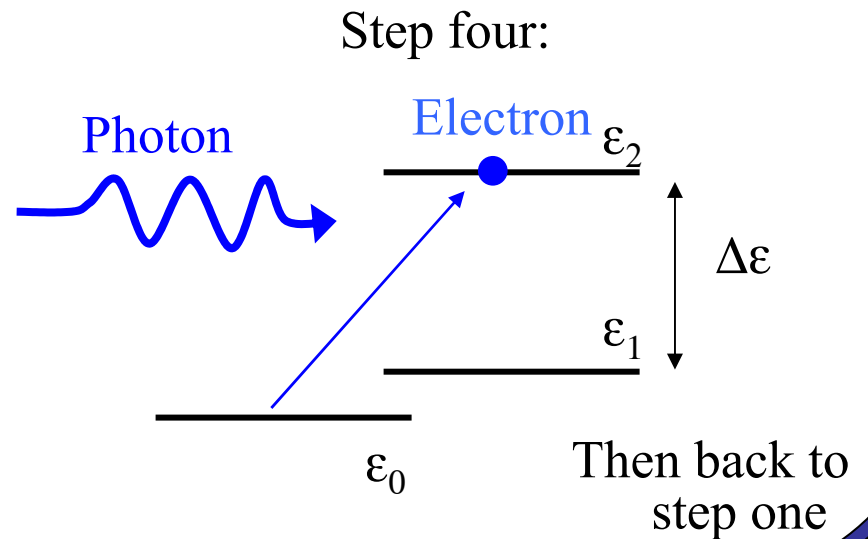
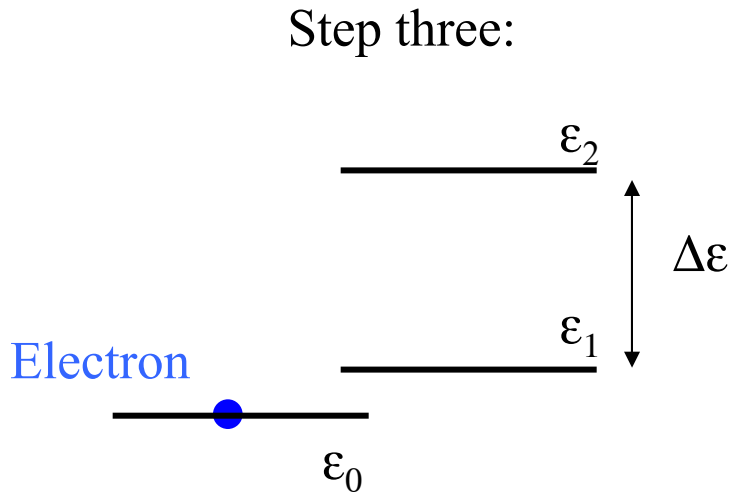
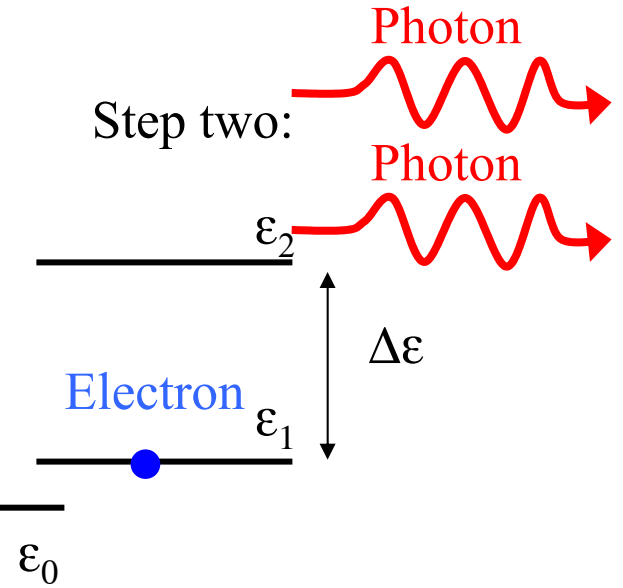
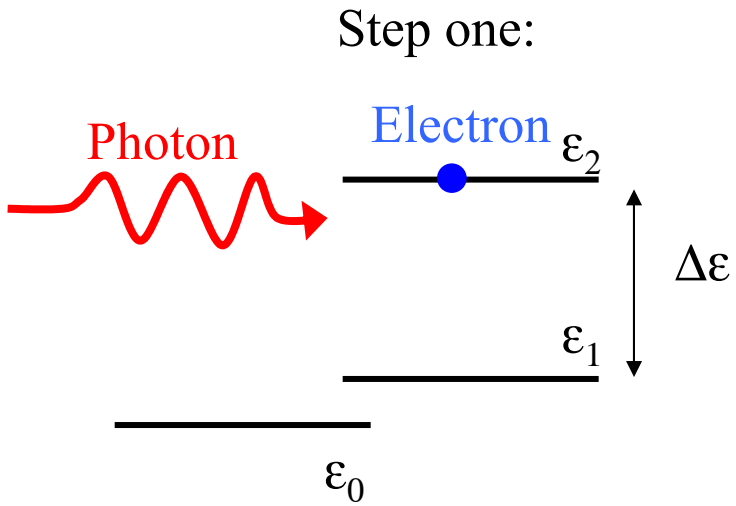
Step two:



Step three:



Three-level-onium:



Three-level-onium:

