

Lecture 2: Review of classical electromagnetic waves

Outline for today: Classical E/M “Bootcamp”

- Maxwell’s equations
- Derivation of the wave equation
- Boundary conditions
- Reflection and refraction
- Ray tracing and conditions of applicability
- Lenses

Reading: Verdeyen chapter 1, inside cover, and your prerequisite electromagnetics course.

Maxwell's equations in vacuum

(Note: Verdeyen left out two equations!)

$$\vec{\nabla} \cdot \vec{e} = \rho / \epsilon_0 \quad \vec{\nabla} \cdot \vec{b} = 0$$

$$\vec{\nabla} \times \vec{e} = -\frac{\partial \vec{b}}{\partial t} \quad \vec{\nabla} \times \vec{b} = \mu_0 \epsilon_0 \frac{\partial \vec{e}}{\partial t} + \mu_0 \vec{j}$$

$$\vec{F} = q(\vec{e} + \vec{v} \times \vec{b})$$

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ F / m}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ H / m}$$

Maxwell's equations: Physical significance

- Particles treated classically: ρ, j
- Fields treated classically: e, b
- Helmholtz theorem:

ANY vector field can be reconstructed from its “div” and “curl”

I.E. if you (the engineer) can set up ρ, j , you can set up e, b !!!

Maxwell's equations: Physical shortcomings

- Matter is not classical, it is quantum
- Matter can only have certain energy levels
- e,b are not classical, they are quantum
- e,b can only have certain energies:
 $E = n h f$
n=number of “photon”
h is Planck's constant
f is the frequency

What if there are no sources?

$$\rho, \mathbf{j} = 0$$

Does that mean $\mathbf{e}, \mathbf{b} = 0$?

What if there are no sources?

$$\rho, \mathbf{j} = 0$$

Does that mean $\mathbf{e}, \mathbf{b} = 0$?

NO!

We can have *waves*.

Why do I care?

Light is a wave.

Wave equation, no sources:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{e}) = \vec{\nabla} \times \left(-\frac{\partial \vec{b}}{\partial t} \right) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{b}) = -\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \vec{e}}{\partial t} \right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{e}}{\partial t^2}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{e}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{e}) - \nabla^2 \vec{e} = -\nabla^2 \vec{e}$$

$$\nabla^2 \vec{e} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{e}}{\partial t^2}$$

Wave equation, no sources:

$$\nabla^2 \vec{e} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{e}}{\partial t^2}$$

$$\nabla^2 \vec{b} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{b}}{\partial t^2}$$

(Homework problem)

Plane waves (draw picture on board):

$$\vec{e}(\vec{r}, t) = \text{Re} \left\{ \left[\vec{E}(\omega, \vec{k}_0) \right] e^{j(\omega t - \vec{k}_0 \cdot \vec{r})} \right\}$$

$$\vec{b}(\vec{r}, t) = \text{Re} \left\{ \left[\vec{B}(\omega, \vec{k}_0) \right] e^{j(\omega t - \vec{k}_0 \cdot \vec{r})} \right\}$$

is a general solution, provided

$$\omega = \frac{1}{\sqrt{\mu_0 \epsilon_0}} k_0 \quad \frac{|E|}{|B|} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Defines the speed of light c :

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s}$$

Superpositions of plane waves:

$$\vec{e}_1(\vec{r}, t) = \text{Re} \left\{ \left[\vec{E}_1(\omega_1, \vec{k}_{01}) \right] e^{j(\omega_1 t - \vec{k}_{01} \cdot \vec{r})} \right\}$$

$$\vec{e}_2(\vec{r}, t) = \text{Re} \left\{ \left[\vec{E}_2(\omega_2, \vec{k}_{02}) \right] e^{j(\omega_2 t - \vec{k}_{02} \cdot \vec{r})} \right\}$$

$$A \vec{e}_1(\vec{r}, t) + B \vec{e}_2(\vec{r}, t)$$

Is also a solution.

In general:

$$\vec{e}(\vec{r}, t) = \sum_n A_n \vec{e}_n(\vec{r}, t)$$

Polarization:

$$\vec{e}(r, t) = \text{Re} \left\{ \left[\vec{E}(\omega, \vec{k}_0) \right] e^{j(\omega t - \vec{k}_0 \cdot \vec{r})} \right\}$$

$$\vec{B}(\omega, \vec{k}_0) = \frac{1}{\omega} \vec{k}_0 \times \vec{E}(\omega, \vec{k}_0)$$

Draw on board: linear, circular polarization.

Poynting vector

$$\vec{S} = c^2 \epsilon_0 \vec{E} \times \vec{B} = \vec{E} \times \vec{H} \quad \text{Watts/m}^2$$

We are most interested in time average:

$$\langle S \rangle_T = \frac{1}{2} c^2 \epsilon_0 E^2 \quad \text{Watts/m}^2$$

Called *intensity* or *irradiance*:
the average energy per unit time per unit area

$$\langle S \rangle_T \cdot \text{Area}$$

Called *optical power* or *radiant flux*

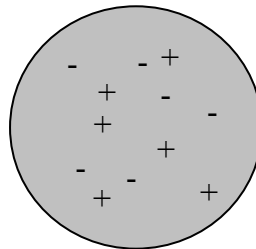
Maxwell's equations in matter

- Why do I care?
- Light passes through matter (glass, etc)
- We need to understand how it propagates in matter.

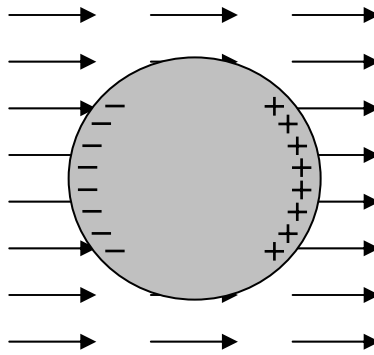
Electric dipole moment

$$\vec{p} = \sum_{i=1}^N q_i \vec{r}_i$$

For neutral matter,
usually 0:



Polarization: Electric field induces dipole moment:



$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

P is dipole moment per unit volume.

This equation defines χ_e

The e is for electric.

Verdeyen does not use the e .

Also works for finite ω .

Polarization:

$$\vec{P} = \chi \epsilon_0 \vec{E}$$

We define displacement vector D :

$$\begin{aligned}\vec{D} &= \epsilon_0 \vec{E} + \vec{P} \\ &= \epsilon_0 \vec{E} + \chi \epsilon_0 \vec{E} \\ &= \epsilon_0 (1 + \chi) \vec{E} \\ &= \epsilon_0 \epsilon_\tau \vec{E}\end{aligned}$$

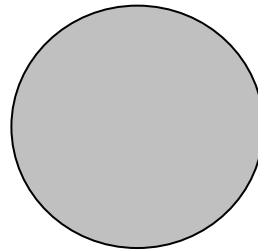
Index of refraction: $n = \sqrt{\epsilon_\tau}$

Wave velocity: $v = c/n$

Magnetic dipole moment

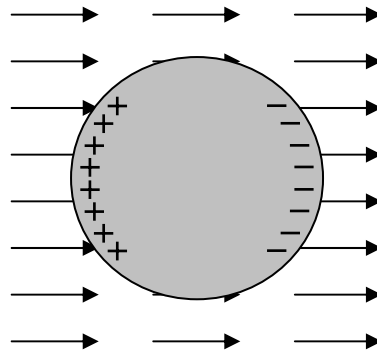
$$\vec{m} = \frac{1}{2} \int_V \vec{r} \times \vec{J}(\vec{r}) dV$$

For neutral matter,
usually 0:



Magnetization:

Magnetic field induces
magnetic dipole moment:



$$\vec{M} = \frac{\chi_m}{\mu_0} \vec{B}$$

M is magnetic dipole moment per unit volume.

This equation defines χ_m
(Not in Verdeyen)

Magnetization:

$$\vec{M} = \frac{\chi_m}{\mu_0} \vec{B}$$

We define field H:

$$\begin{aligned} \vec{H} &= \frac{1}{\mu_0} \vec{B} - \vec{M} \\ &= \frac{1}{\mu_0} \vec{B} - \frac{\chi_m}{\mu_0} \vec{B} \\ &= \vec{B} \frac{1}{\mu_0} (1 - \chi_m) = \vec{B} \frac{1}{\mu_r \mu_0} \end{aligned}$$

Index of refraction: $n = \sqrt{\epsilon_r \mu_r}$

Wave velocity: $v = c/n$

Maxwell's equations in solids

$$\vec{\nabla} \cdot \vec{d} = \rho_f$$

$$\vec{\nabla} \cdot \vec{b} = 0$$

$$\vec{\nabla} \times \vec{e} = -\frac{\partial \vec{b}}{\partial t}$$

$$\vec{\nabla} \times \vec{h} = \frac{\partial \vec{d}}{\partial t} + \vec{j}_f$$

The subscript f is for “free”, i.e. those charges that do not participate in the magnetic or electric susceptibility.

Some people think this form of Maxwell's equations is “simpler”.

Characteristic impedance:

In vacuum:

$$\frac{|E|}{|B|} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c \qquad \frac{|E|}{|H|} = \sqrt{\frac{\mu_0}{\epsilon_0}} = Z_0 = 377\Omega$$

In dielectric:

$$\frac{|E|}{|B|} = \frac{1}{\sqrt{\mu_0 \mu_\tau \epsilon_0 \epsilon_\tau}} = \frac{c}{n} \qquad \frac{|E|}{|H|} = \sqrt{\frac{\mu_0 \mu_\tau}{\epsilon_0 \epsilon_\tau}} = Z$$

These are important in antenna and transmission line problems where we want to impedance match a circuit to a plane wave.

Boundary conditions:

$$\vec{a}_n \times (\vec{E}_1 - \vec{E}_2) = 0 \quad \Leftrightarrow \quad \vec{E}_{\parallel 2} = \vec{E}_{\parallel 1}$$

$$\vec{a}_n \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \quad \Leftrightarrow \quad \vec{D}_{\perp 1} - \vec{D}_{\perp 2} = \rho_s$$

$$\vec{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \quad \Leftrightarrow \quad \vec{H}_{\parallel 2} - \vec{H}_{\parallel 1} = \vec{J}_s$$

$$\vec{a}_n \cdot (\vec{B}_1 - \vec{B}_2) = 0 \quad \Leftrightarrow \quad \vec{B}_{\perp 1} = \vec{B}_{\perp 2}$$

• You will prove in HW, starting from Maxwell's equations

Why do I care?

Important to understand reflection and transmission of light.

Example proof:

$$\vec{a}_n \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$$

We will prove for $\rho_s = 0$.

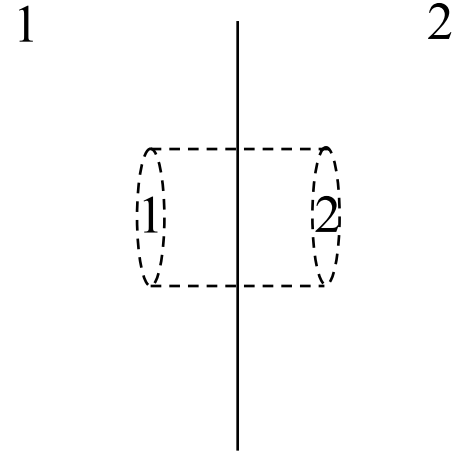
$$\vec{\nabla} \cdot \vec{d} = 0$$

$$\iiint_V dV \vec{\nabla} \cdot \vec{d} = \oiint_S dA \vec{d} \cdot \vec{a}_n$$

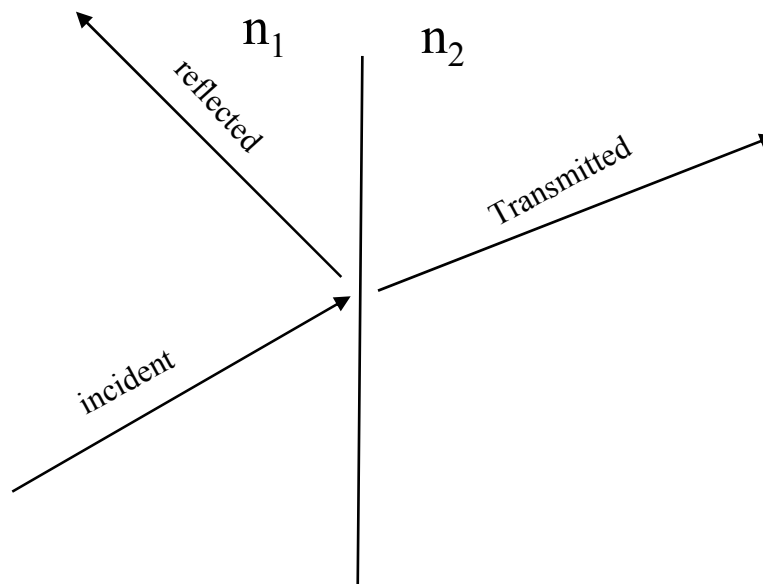
||
0

$$= \vec{d}_1 \cdot \Delta \vec{a}_1 + \vec{d}_2 \cdot \Delta \vec{a}_2 + \text{circ.term}$$

$$= \vec{d}_1 \cdot \Delta \vec{a}_1 + \vec{d}_2 \cdot (-\Delta \vec{a}_1) = \vec{a}_n \cdot (\vec{d}_1 - \vec{d}_2)$$



Reflection and refraction:

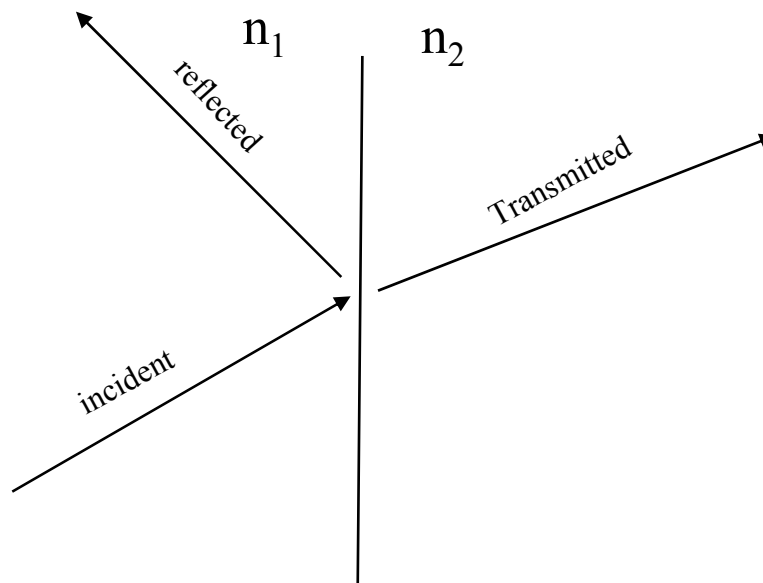


Why do I care?

This is the simplest geometry for reflection and transmission.

This will form the basis of understanding mirrors that form a laser cavity.

Reflection and refraction:



$$\vec{E}_i = \vec{E}_{0i} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

$$\vec{E}_t = \vec{E}_{0t} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

$$\vec{E}_r = \vec{E}_{0r} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

(Write on board)

Reflection and refraction:

$$k_i^2 = \left(\frac{\omega_i}{v_1}\right)^2 = \left(\frac{n_1\omega_i}{c}\right)^2$$

$$k_t^2 = \left(\frac{\omega_t}{v_2}\right)^2 = \left(\frac{n_2\omega_t}{c}\right)^2$$

$$k_r^2 = \left(\frac{\omega_r}{v_1}\right)^2 = \left(\frac{n_1\omega_r}{c}\right)^2$$

Boundary condition:

E_{\parallel} is continuous

$$\left[\vec{E}_{0i} e^{i(\vec{k}_i \cdot \vec{r}_B - \omega t)} + \vec{E}_{0r} e^{i(\vec{k}_r \cdot \vec{r}_B - \omega t)} \right]_{\parallel} = \left[\vec{E}_{0t} e^{i(\vec{k}_t \cdot \vec{r}_B - \omega t)} \right]_{\parallel}$$

So: $\omega_i = \omega_r = \omega_t = \omega$

$$k_i^2 = k_r^2 = \left(\frac{n_1\omega}{c}\right)^2 = k_1^2 \quad k_t^2 = \left(\frac{n_2\omega}{c}\right)^2 = k_2^2$$

What about direction of \mathbf{k} ?

$$\vec{k}_i \cdot \vec{r}_B = \vec{k}_t \cdot \vec{r}_B = \vec{k}_r \cdot \vec{r}_B$$

We will explore these equations
one by one.

What about direction of k ?

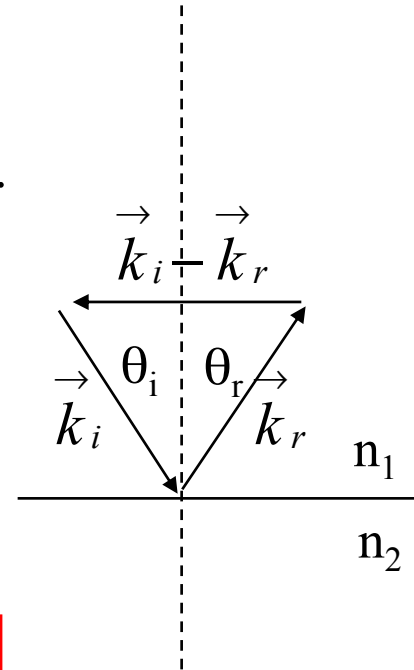
$$\left(\vec{k}_i - \vec{k}_r \right) \cdot \vec{r}_B = 0$$

Let the origin be on the surface.

Then the direction of \vec{r}_B is always parallel to the surface.

(Draw dot product geometry on board).

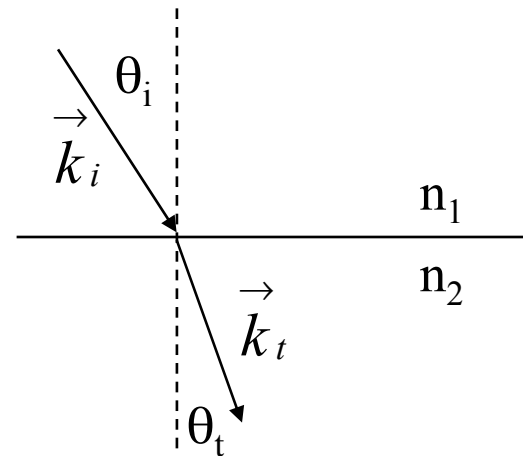
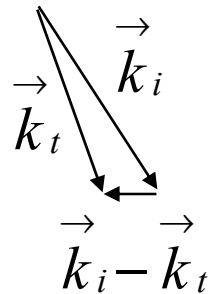
Then this says the component of $\left(\vec{k}_i - \vec{k}_r \right)$ perpendicular to the surface is zero.



$$\theta_i = \theta_r$$

What about direction of k ?

$$\left(\vec{k}_i - \vec{k}_t \right) \cdot \vec{r}_B = 0$$



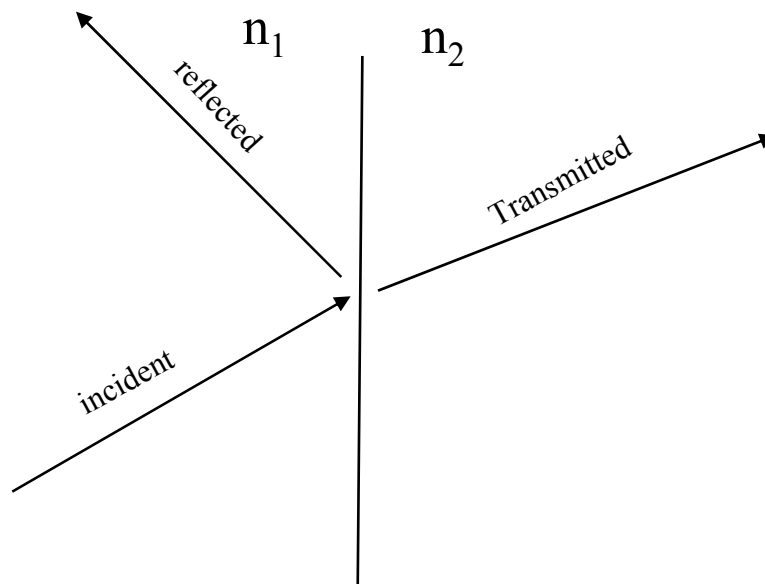
$$k_i \sin(\theta_i) = k_t \sin(\theta_t)$$

$$k_i = (n_1 \omega / c) \quad k_t = (n_2 \omega / c)$$

$$n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$$

Snell's law

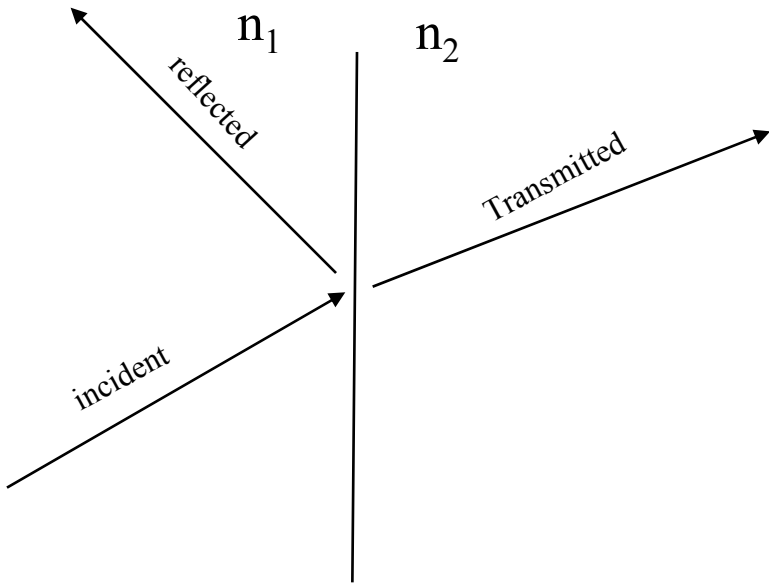
Reflection and refraction:



Now we know about direction of reflected and transmitted waves.

What about the amplitude?

Reflection and refraction:



$$\vec{E}_i = \vec{E}_{0i} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

$$\vec{E}_r = \vec{E}_{0r} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

$$\vec{E}_t = \vec{E}_{0t} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

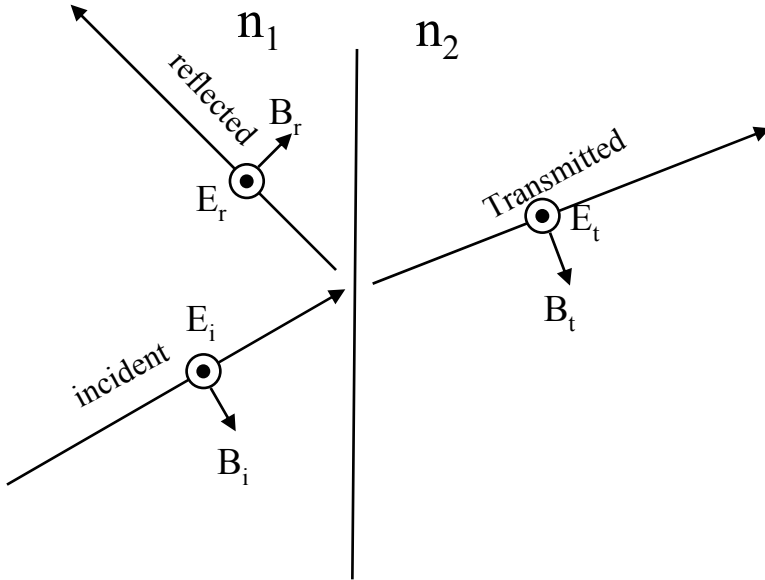
We want to know:

$$\frac{E_r}{E_i} = ???$$

$$\frac{E_t}{E_i} = ???$$

Results are the *Fresnel Equations*.

Fresnel equations: E_{\perp} :



$$\vec{E}_i = \vec{E}_{0i} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

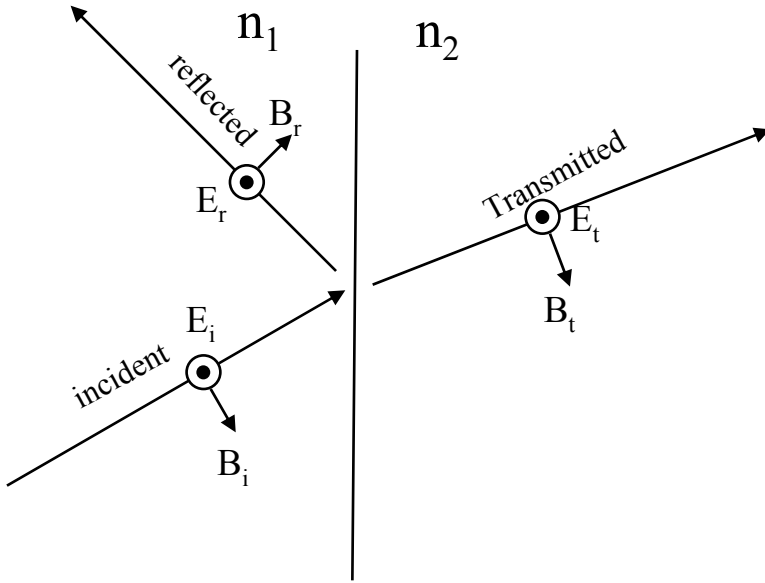
$$\vec{E}_r = \vec{E}_{0r} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

$$\vec{E}_t = \vec{E}_{0t} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

Continuity in $E_{\parallel} \Rightarrow$

$$E_{0i} + E_{0r} = E_{0t}$$

Fresnel equations: E_{\perp} :



$$\vec{E}_i = \vec{E}_{0i} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

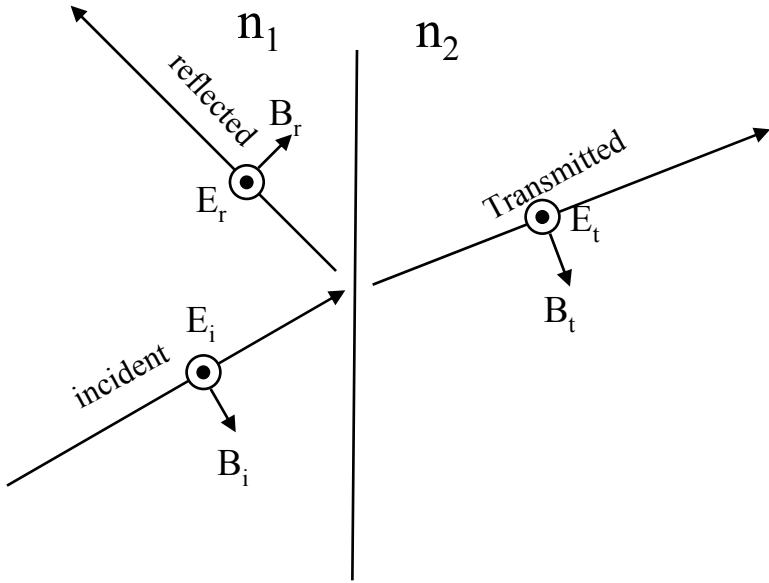
$$\vec{E}_r = \vec{E}_{0r} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

$$\vec{E}_t = \vec{E}_{0t} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

Continuity in $E_{\parallel} \Rightarrow E_{0i} + E_{0r} = E_{0t}$

Continuity in $H_{\parallel} \Rightarrow (H_{0i})_{\parallel} + (H_{0r})_{\parallel} = (H_{0t})_{\parallel}$

Fresnel equations: E_{\perp} :



$$\vec{E}_i = \vec{E}_{0i} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

$$\vec{E}_r = \vec{E}_{0r} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

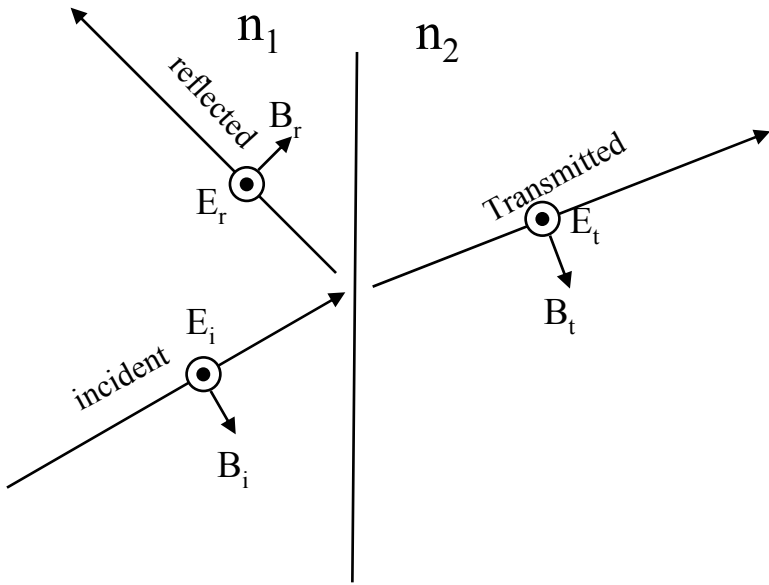
$$\vec{E}_t = \vec{E}_{0t} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

Continuity in $E_{\parallel} \Rightarrow E_{0i} + E_{0r} = E_{0t}$

Continuity in $H_{\parallel} \Rightarrow (H_{0i})_{\parallel} + (H_{0r})_{\parallel} = (H_{0t})_{\parallel}$

$$-\frac{B_{0i}}{\mu_1} \cos(\theta_i) + -\frac{B_{0r}}{\mu_1} \cos(\theta_r) = \frac{B_{0t}}{\mu_2} \cos(\theta_t)$$

Fresnel equations: E_{\perp} :



$$\vec{E}_i = \vec{E}_{0i} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

$$\vec{E}_r = \vec{E}_{0r} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

$$\vec{E}_t = \vec{E}_{0t} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

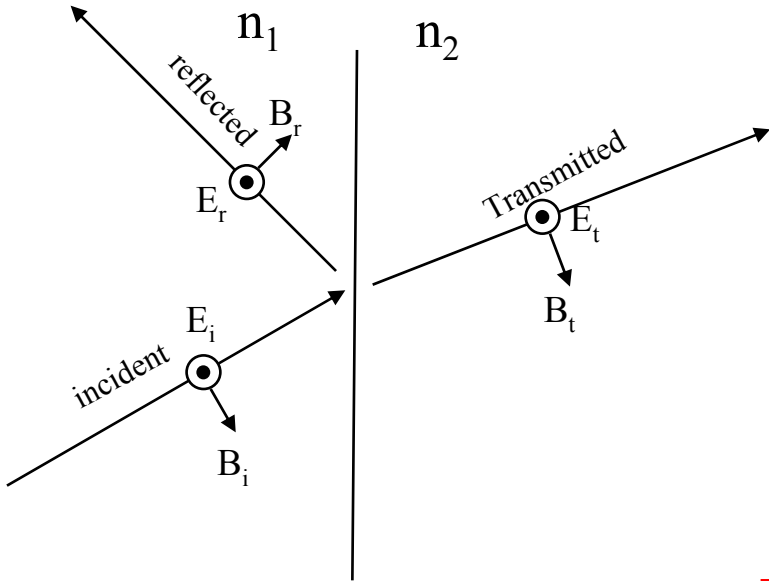
Continuity in $E_{\parallel} \Rightarrow E_{0i} + E_{0r} = E_{0t}$

Continuity in $H_{\parallel} \Rightarrow (H_{0i})_{\parallel} + (H_{0r})_{\parallel} = (H_{0t})_{\parallel}$

$$-\frac{B_{0i}}{\mu_1} \cos(\theta_i) + -\frac{B_{0r}}{\mu_1} \cos(\theta_r) = \frac{B_{0t}}{\mu_2} \cos(\theta_t)$$

$$-\frac{E_{0i} n_1}{c \mu_1} \cos(\theta_i) + -\frac{E_{0r} n_1}{c \mu_1} \cos(\theta_r) = \frac{E_{0t} n_2}{c \mu_2} \cos(\theta_t)$$

Fresnel equations: E_{\perp}



$$\vec{E}_i = \vec{E}_{0i} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

$$\vec{E}_r = \vec{E}_{0r} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

$$\vec{E}_t = \vec{E}_{0t} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

Continuity in $E_{\parallel} \Rightarrow$

$$E_{0i} + E_{0r} = E_{0t}$$

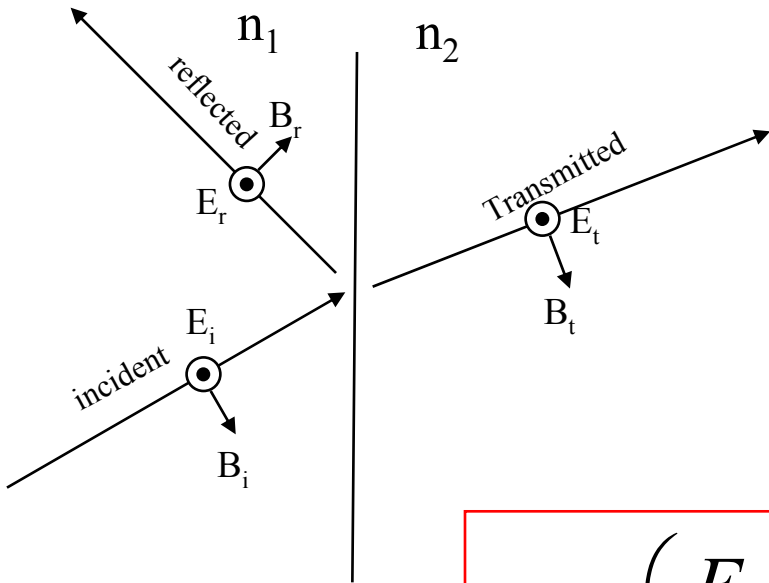
Continuity in $H_{\parallel} \Rightarrow$

$$(H_{0i})_{\parallel} + (H_{0r})_{\parallel} = (H_{0t})_{\parallel}$$

$$-\frac{B_{0i}}{\mu_1} \cos(\theta_i) + -\frac{B_{0r}}{\mu_1} \cos(\theta_r) = \frac{B_{0t}}{\mu_2} \cos(\theta_t)$$

$$-\frac{E_{0i} n_1}{c \mu_1} \cos(\theta_i) + -\frac{E_{0r} n_1}{c \mu_1} \cos(\theta_r) = \frac{E_{0t} n_2}{c \mu_2} \cos(\theta_t)$$

Fresnel equations: E_{\perp}



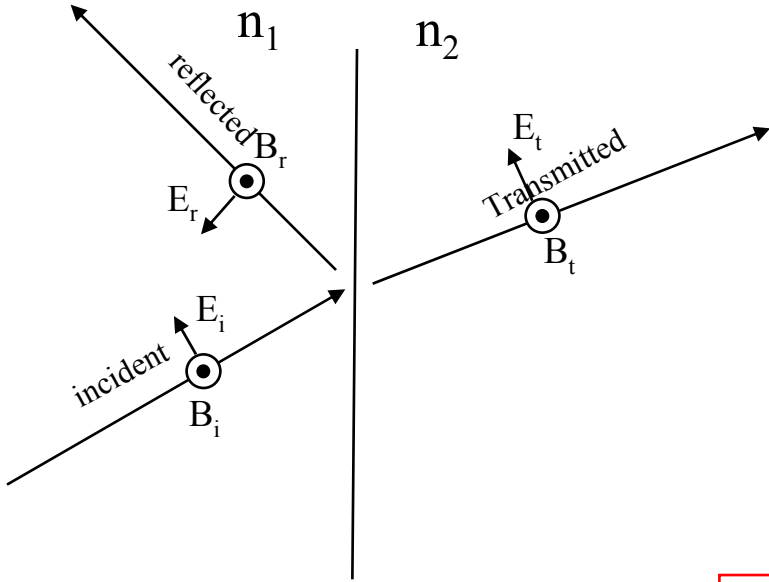
Amplitude
reflection
coefficients:

$$r_{\perp} \equiv \left(\frac{E_{0r}}{E_{0i}} \right)_{\perp} = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2}$$

$$t_{\perp} \equiv \left(\frac{E_{0r}}{E_{0i}} \right)_{\perp} = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + n_2 \cos \theta_2}$$

(Write normal incidence formula on the board.)

Fresnel equations: E_{\parallel}



$$\vec{E}_i = \vec{E}_{0i} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

$$\vec{E}_r = \vec{E}_{0r} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

$$\vec{E}_t = \vec{E}_{0t} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

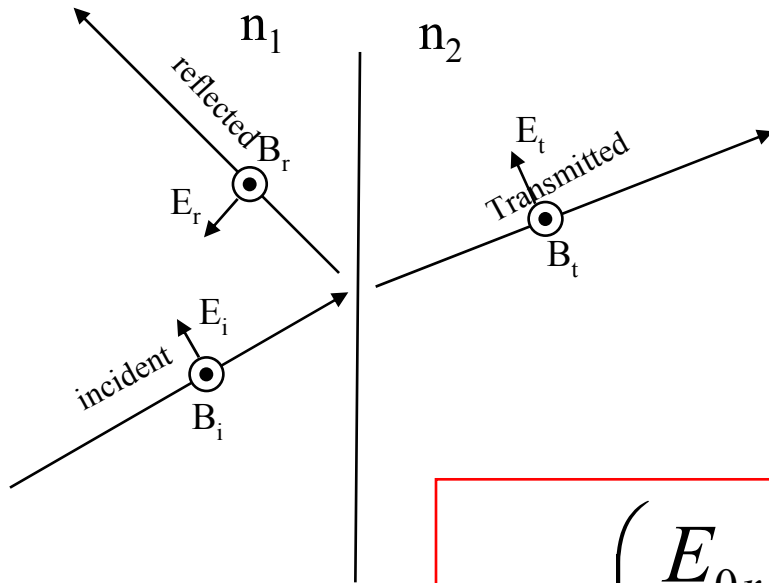
Continuity in $E_{\parallel} \Rightarrow$

$$E_{0i} \cos \theta_i - E_{0r} \cos \theta_r = E_{0t} \cos \theta_t$$

Continuity in $H_{\parallel} \Rightarrow$

$$\frac{n_1}{\mu_1 c} E_{0i} + \frac{n_1}{\mu_1 c} E_{0r} = \frac{n_2}{\mu_2 c} E_{0t}$$

Fresnel equations: E_{\parallel}



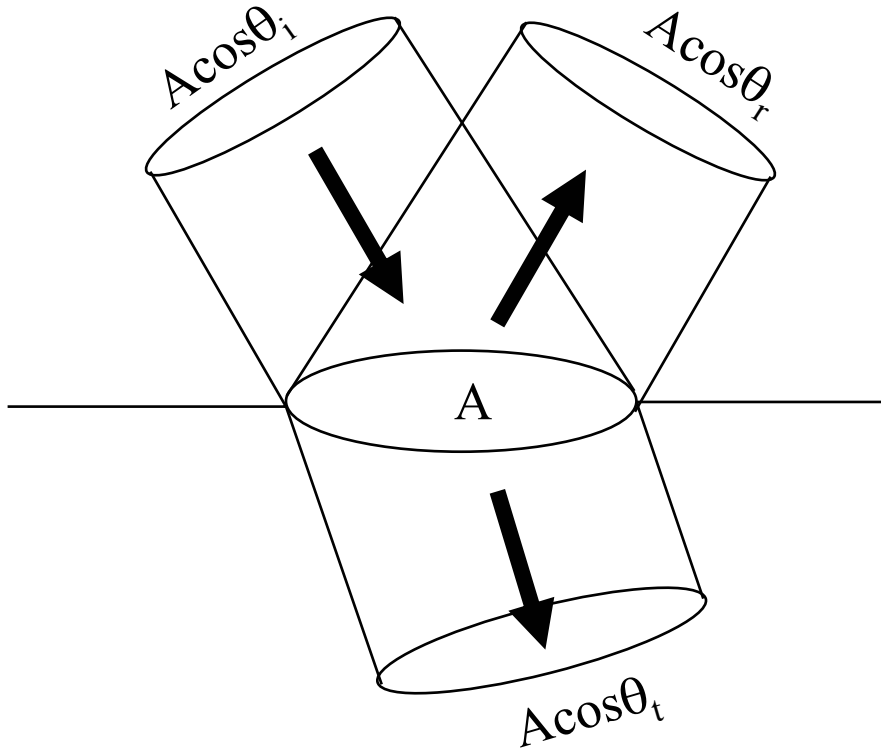
Amplitude
reflection
coefficients:

$$r_{\parallel} \equiv \left(\frac{E_{0r}}{E_{0i}} \right)_{\parallel} = \frac{n_2 \cos \theta_1 - n_1 \cos \theta_2}{n_2 \cos \theta_1 + n_1 \cos \theta_2}$$

$$t_{\parallel} \equiv \left(\frac{E_{0r}}{E_{0i}} \right)_{\parallel} = \frac{2n_1 \cos \theta_1}{n_2 \cos \theta_1 + n_1 \cos \theta_2}$$

(Write normal incidence formula on the board.)

Reflectance and Transmittance



Reflectance=
reflected power/incident power

$$R \equiv \frac{I_r A \cos \theta_r}{I_i A \cos \theta_i} = \frac{I_r}{I_i} = \left(\frac{E_{0r}}{E_{0i}} \right)^2 = r^2$$

Transmittance=
transmitted power/incident power

$$T \equiv \frac{I_t A \cos \theta_t}{I_i A \cos \theta_i} = \left(\frac{n_t \cos \theta_t}{n_i \cos \theta_i} \right)^2 t^2$$

$$R + T = 1$$

GaAs laser cleaved facet

(HW allows you to study the electromagnetic properties of the
cavity without a gain medium.)

$$n=3.5$$



What does gain need to
be for lasing?

Total internal reflection

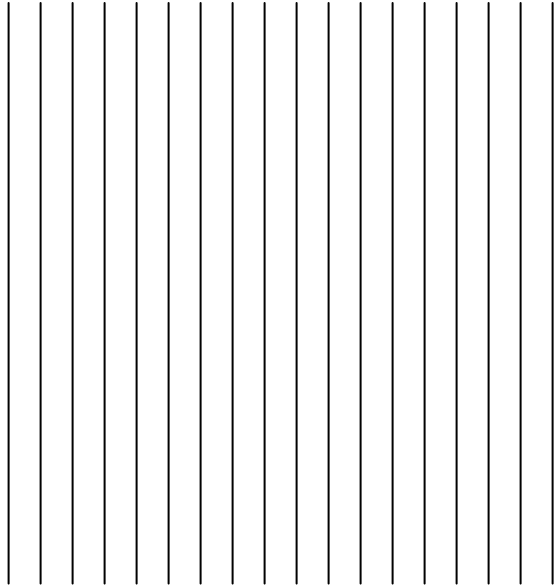
$$n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$$

Snell's law

(Discuss on board)

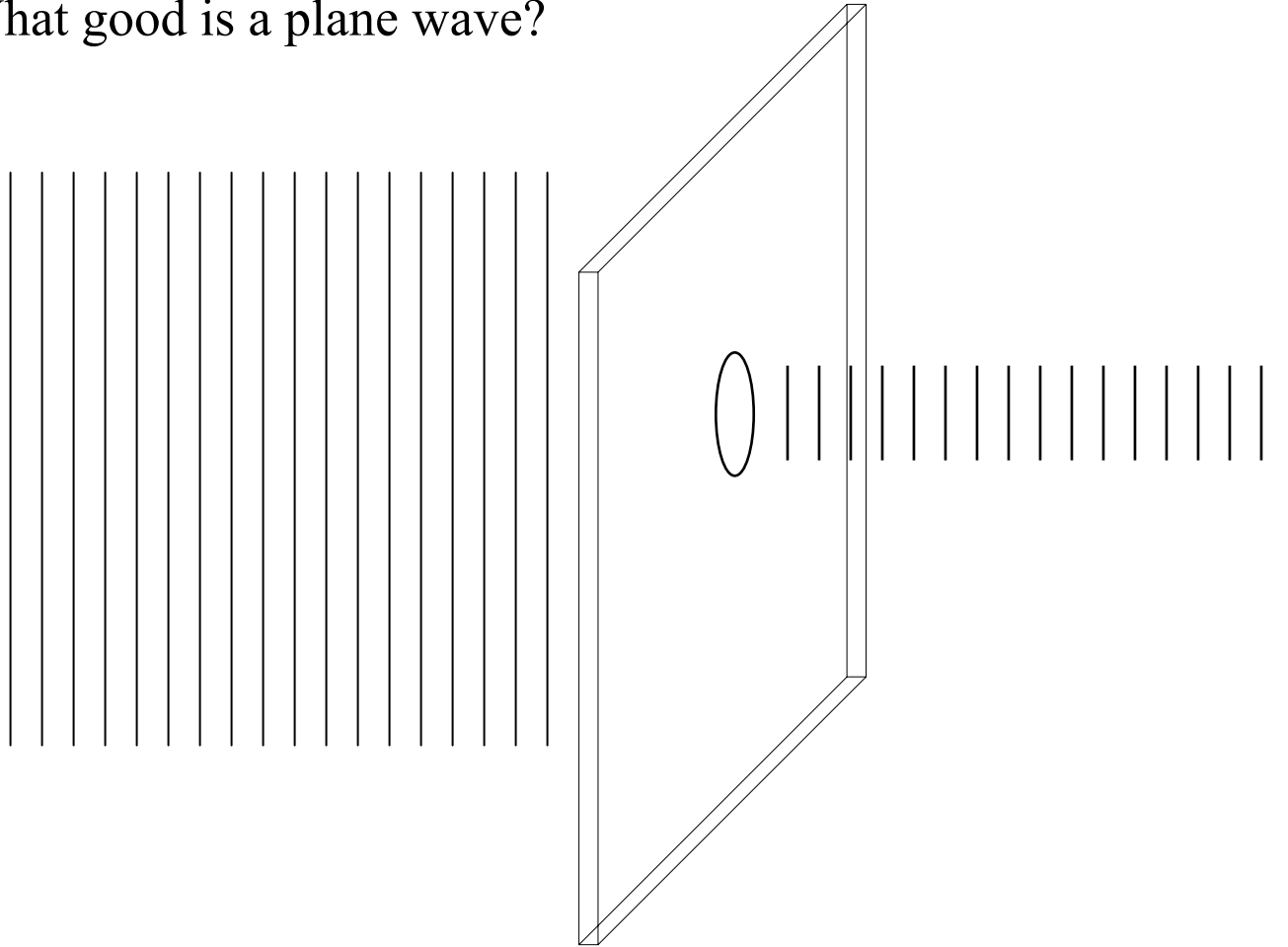
Ray tracing (geometrical optics)

What good is a plane wave?



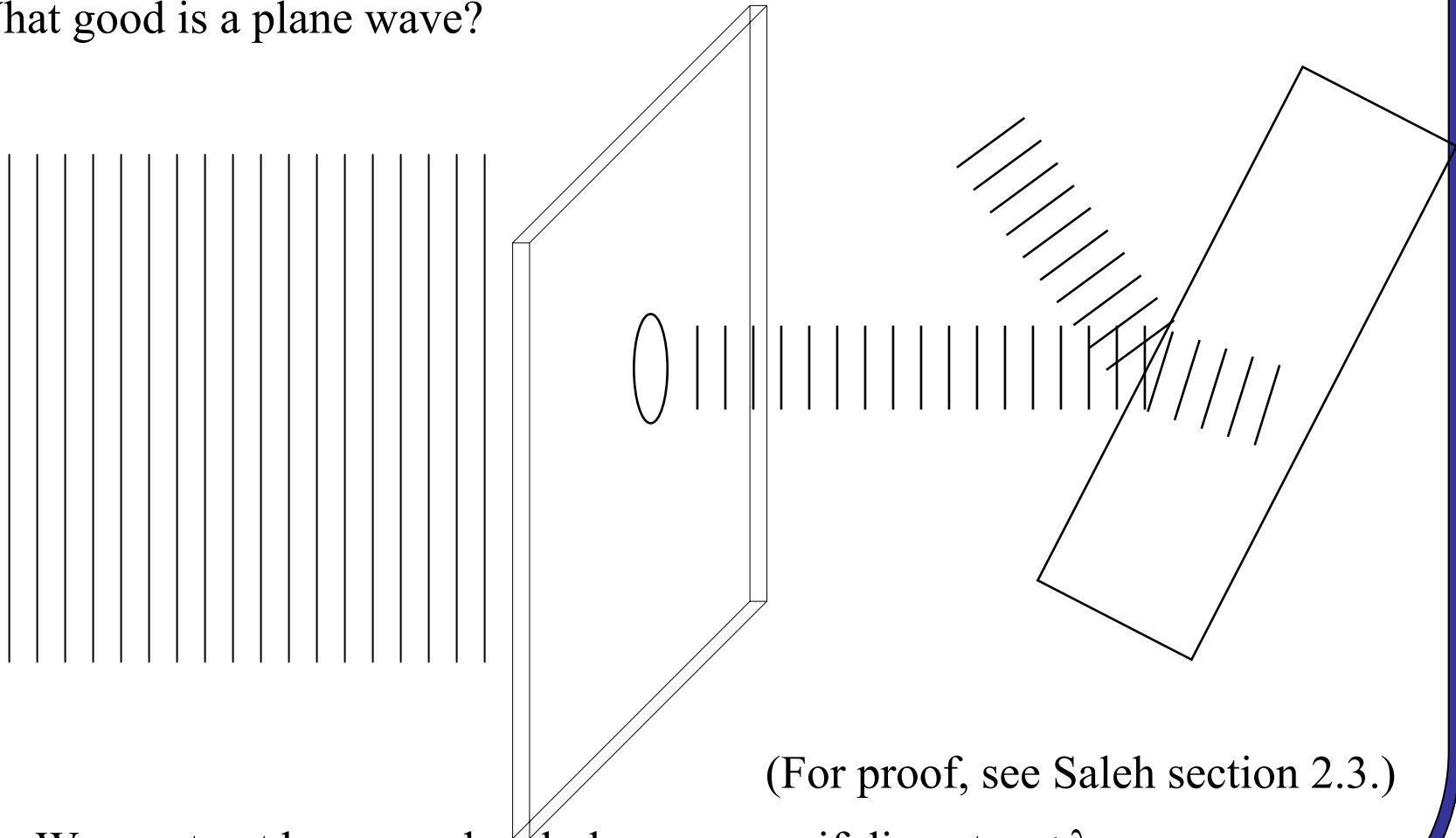
Ray tracing (geometrical optics)

What good is a plane wave?



Ray tracing (geometrical optics)

What good is a plane wave?



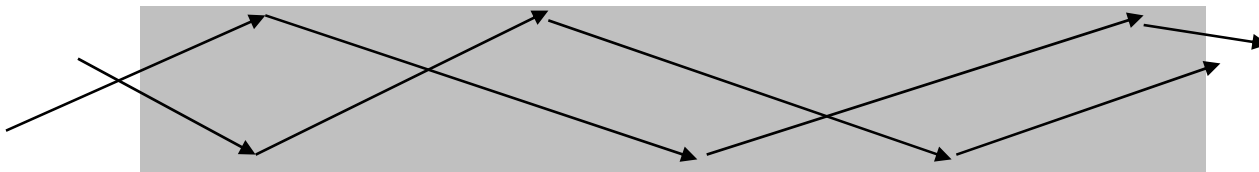
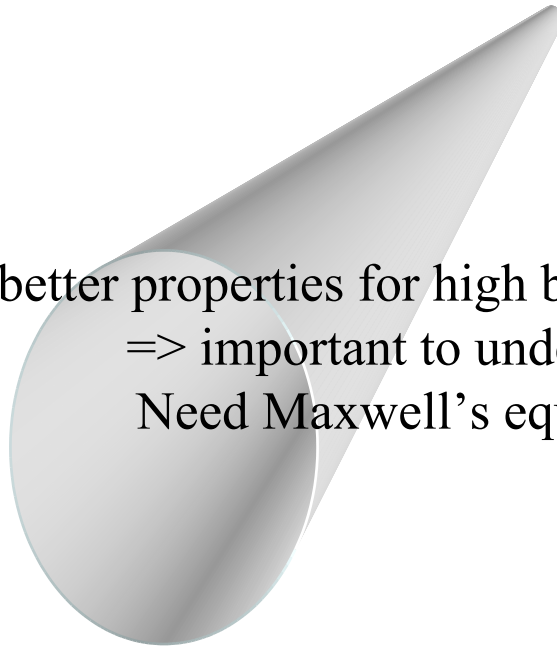
(For proof, see Saleh section 2.3.)

We can treat beams as local plane waves, if diameter $< \lambda$.

Otherwise, need to go back to Maxwell's equations! See lectures 5,6.

Example: optical fiber

Small diameter has better properties for high bit rate, long-haul communications
=> important to understand.
Need Maxwell's equations!



This description of fibers works only if diameter $> \lambda$.
Not always true!

Example: thick lens equation

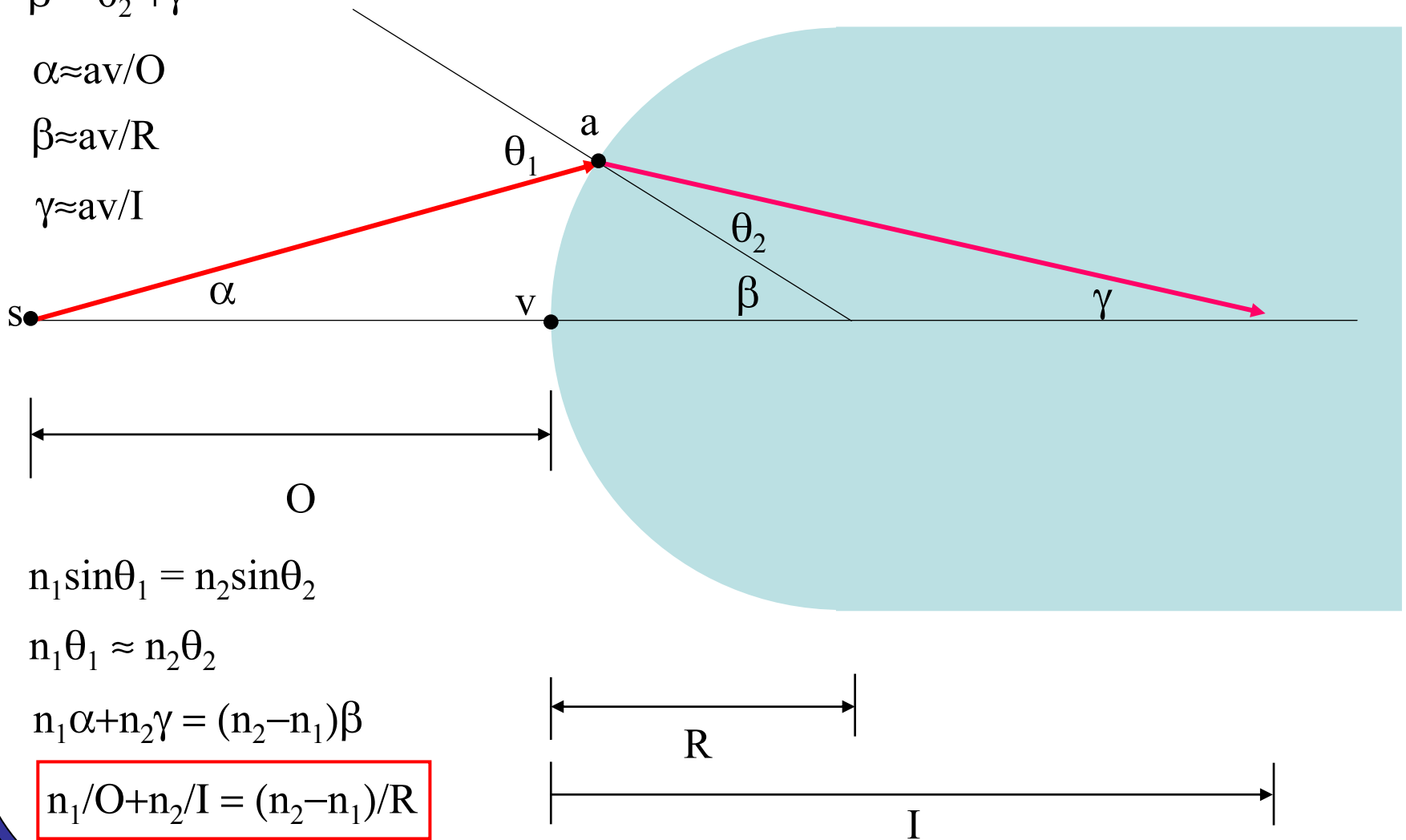
$$\theta_1 = \alpha + \gamma$$

$$\beta = \theta_2 + \gamma$$

$$\alpha \approx av/O$$

$$\beta \approx av/R$$

$$\gamma \approx av/I$$



$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

$$n_1 \theta_1 \approx n_2 \theta_2$$

$$n_1 \alpha + n_2 \gamma = (n_2 - n_1) \beta$$

$$n_1/O + n_2/I = (n_2 - n_1)/R$$

All of geometrical
optics treated similarly.

Next week:

Photons Atoms