HW6 problem 1a

$$\frac{e(t)}{E_0} = \sum_{-(N-1)/2}^{(N-1)/2} e^{j\omega_o t} e^{jn\omega_c t} = e^{j\omega_o t} \sum_{-(N-1)/2}^{(N-1)/2} \left(e^{j\omega_c t}\right)^n = e^{j\omega_o t} \sum_{m'=0}^{2\left(\frac{N-1}{2}\right)} (e^{j\omega_c t})^{m'-(N-1)/2}$$

$$= e^{j\omega_o t} (e^{j\omega_c t})^{-(N-1)/2} \sum_{m'=0}^{2\left(\frac{N-1}{2}\right)} (e^{j\omega_c t})^{m'} = e^{j\omega_o t} (e^{j\omega_c t})^{-(N-1)/2} \frac{1-(e^{j\omega_c t})^N}{1-e^{j\omega_c t}} =$$

$$= e^{j\omega_o t} (e^{j\omega_c t})^{1/2} \frac{(e^{j\omega_c t})^{-N/2} - (e^{j\omega_c t})^{-N/2} (e^{j\omega_c t})^N}{1-e^{j\omega_c t}}$$

$$= e^{j\omega_o t} (e^{j\omega_c t})^{1/2} \frac{(e^{j\omega_c t})^{-N/2} - (e^{j\omega_c t})^{-N/2} (e^{j\omega_c t})^N}{1-e^{j\omega_c t}}$$

$$= e^{j\omega_o t} \frac{(e^{j\omega_c t})^{-N/2} - (e^{j\omega_c t})^{+N/2}}{(e^{j\omega_c t})^{-1/2} - (e^{j\omega_c t})^{-1/2} e^{j\omega_c t}}$$

$$= e^{j\omega_o t} \frac{e^{jN\omega_c t/2} - e^{+jN\omega_c t/2}}{e^{-j\omega_c t/2} - e^{+j\omega_c t/2}} = e^{j\omega_o t} \frac{\sin(N\omega_c t/2)}{\sin(\omega_c t/2)}$$

HW6 problem 1b

B) Free spectral range = c/2L = 150 MHz => 10 modes would lase.

C) See plots.

For part C: the time between pulses is always 2L/c, and the pulse width is always ~ 1/(gain bandwidth)

D) Free spectral range = c/2L = 150 MHz => 800 modes would lase

E) See plots

F) YAG gives shorter pulses, because gain-bandwidth is larger.

HW6 problem 2

$$\sin(a+b)=\sin(a)\cos(b)+\cos(b)\sin(a)$$

$$\omega_c=\pi/L$$

$$\tau_{RT}=2L/c$$

$$\omega_c\tau_{RT}=2\pi$$

$$\frac{e(t)}{E_0}=e^{j\omega_o t}\frac{\sin(N\omega_c t/2)}{\sin(\omega_c t/2)}$$

$$\frac{e(t+\tau_{RT})}{E_0}=e^{j\omega_o(t+\tau_{RT})}\frac{\sin(N\omega_c(t+\tau_{RT})/2)}{\sin(\omega_c(t+\tau_{RT})/2)}$$

$$=e^{j\omega_o(t+\tau_{RT})}\frac{\sin(N\omega_c t/2)\cos(N\omega_c\tau_{RT}/2)+\cos(N\omega_c t/2)\sin(N\omega_c\tau_{RT}/2)}{\sin(\omega_c t/2)\cos(\omega_c\tau_{RT}/2)+\cos(\omega_c t/2)\sin(\omega_c\tau_{RT}/2)}$$

$$=e^{j\omega_o(t+\tau_{RT})}\frac{\sin(N\omega_c t/2)\cos(N\pi)+\cos(N\omega_c t/2)\sin(N\pi)}{\sin(\omega_c t/2)\cos(\pi)+\cos(\omega_c t/2)\sin(\pi)}$$

$$=e^{j\omega_o(t+\tau_{RT})}\frac{\sin(N\omega_c t/2)\cos(N\pi)+\cos(N\omega_c t/2)\sin(N\pi)}{\sin(\omega_c t/2)\cos(\pi)+\cos(\omega_c t/2)\sin(\pi)}$$

$$=e^{j\omega_o(t+\tau_{RT})}\frac{\sin(N\omega_c t/2)\cos(N\pi)}{\sin(\omega_c t/2)\cos(\pi)}$$

$$=e^{j\omega_o(t+\tau_{RT})}\frac{\sin(N\omega_c t/2)}{\sin(\omega_c t/2)}(-1)^N$$

$$=e^{jn\omega_c(t+\tau_{RT})}\frac{\sin(N\omega_c t/2)}{\sin(\omega_c t/2)}(-1)^N$$

$$=e^{j\omega_o t}e^{jn\omega_c\tau_{RT}}\frac{\sin(N\omega_c t/2)}{\sin(\omega_c t/2)}(-1)^N$$

$$=e^{j\omega_o t}e^{jn2\pi}\frac{\sin(N\omega_c t/2)}{\sin(\omega_c t/2)}(-1)^N$$

$$=e^{j\omega_o t}\frac{\sin(N\omega_c t/2)}{\sin(\omega_c t/2)}(-1)^N=\frac{e(t)}{E_0}$$

Up to a sign.

HW6 problem 3

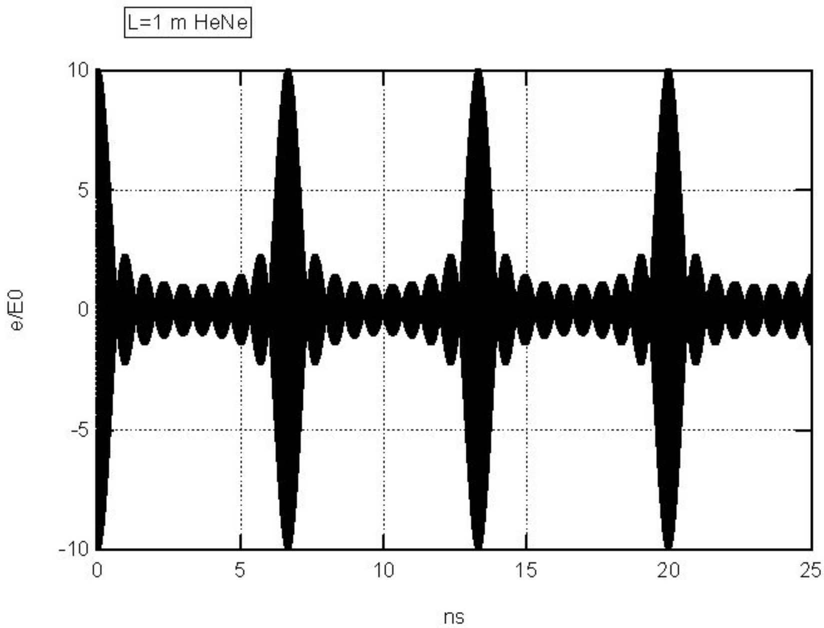
See scanned solution. But I quote from the book:

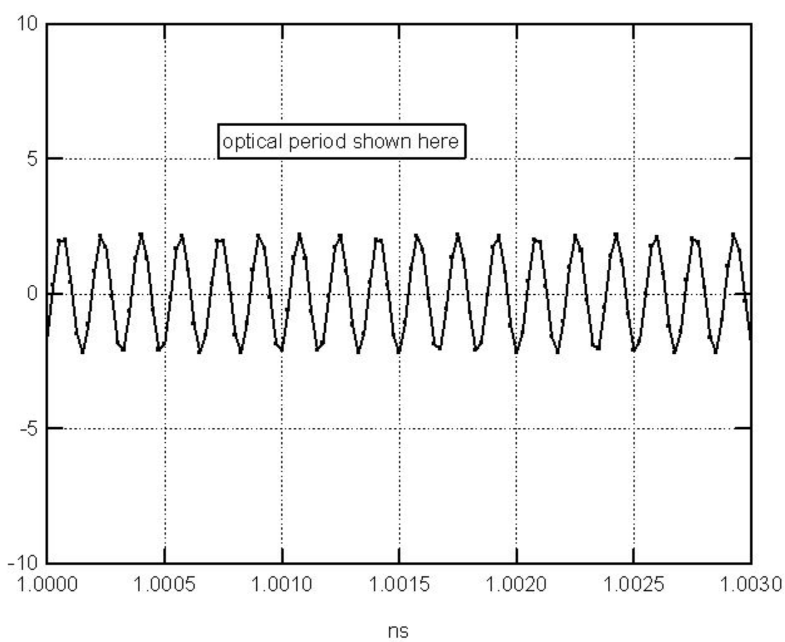
"The first and most important step (and the one usually bypassed by the uninitiated) is to use the information on spectral distribution to establish an equation for the electric field (not the intensity)…"

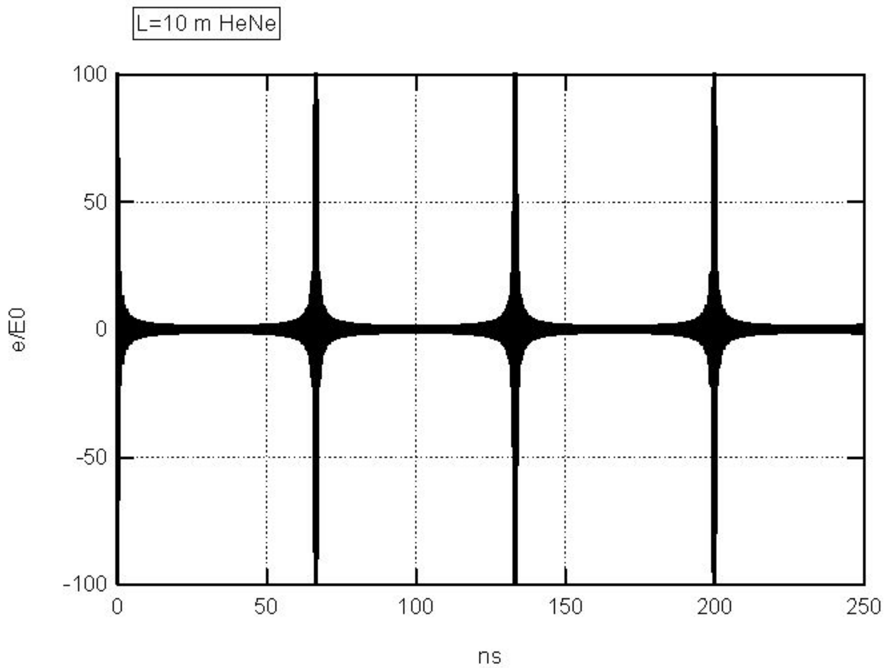
HW6 problem 4

$$\frac{\partial^2}{\partial t^2}\delta I(t)+\frac{\sigma R_{2,ss}}{\alpha}\frac{\partial}{\partial t}\delta I(t)+\frac{\sigma I_{ss} c\alpha}{h\upsilon}\delta I(t)=\left(c\sigma I_{ss}\right)\delta R_2(t)$$

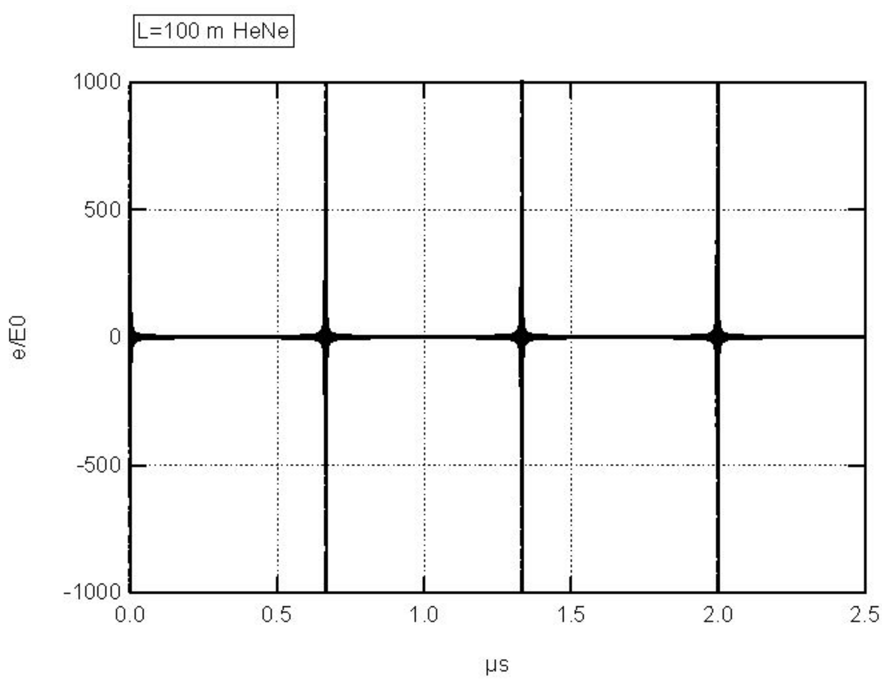
$$\frac{\delta I(\omega)}{\delta R_2(\omega)}=\frac{c\sigma I_{ss}}{\frac{\sigma I_{ss} c\alpha}{h\upsilon}-\omega^2+j\omega\frac{\sigma R_{2,ss}}{\alpha}}$$


L=1 m HeNe
10
5
0
-5
-10
e/E0
0
5
10
15
20
25
ns

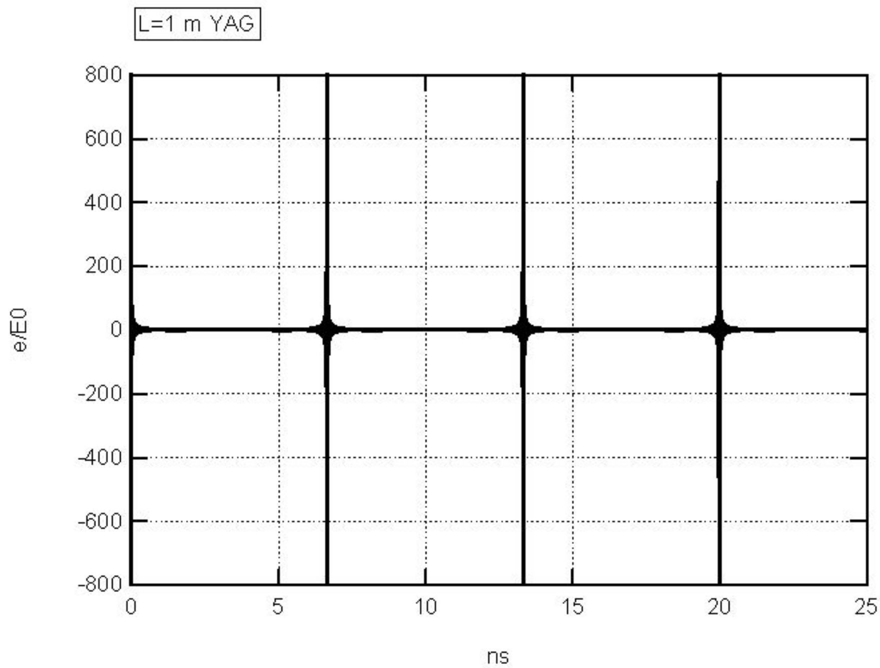

10
5
0
-5
-10
optical period shown here
1.0000
1.0005
1.0010
1.0015
1.0020
1.0025
1.0030
ns



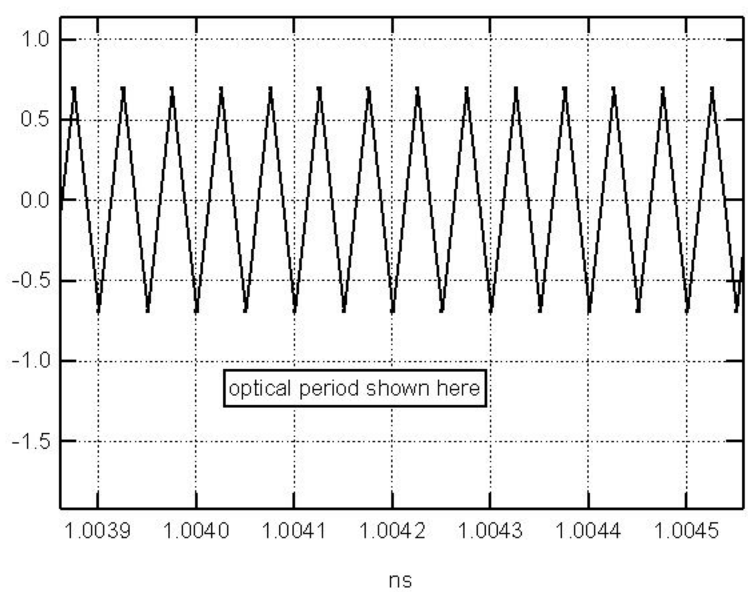
L=10 m HeNe
100
50
0
-50
-100
0
50
100
150
200
250
ns
e/E0



L=100 m HeNe
1000
500
0
-500
-1000
0.0
0.5
1.0
1.5
2.0
2.5
µs
e/E0

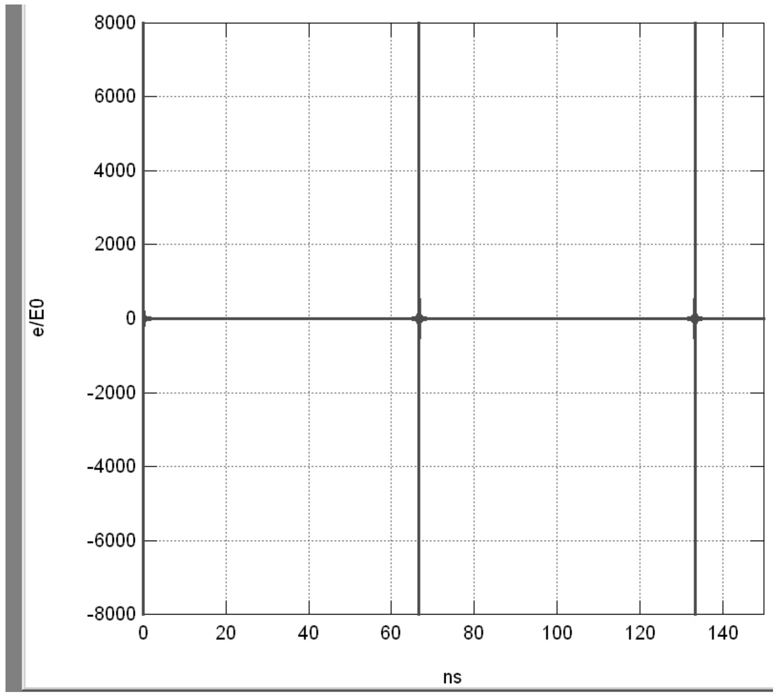


L=1 m YAG
800
600
400
200
0
-200
-400
-600
-800
e/E0
0
5
10
15
20
25
ns

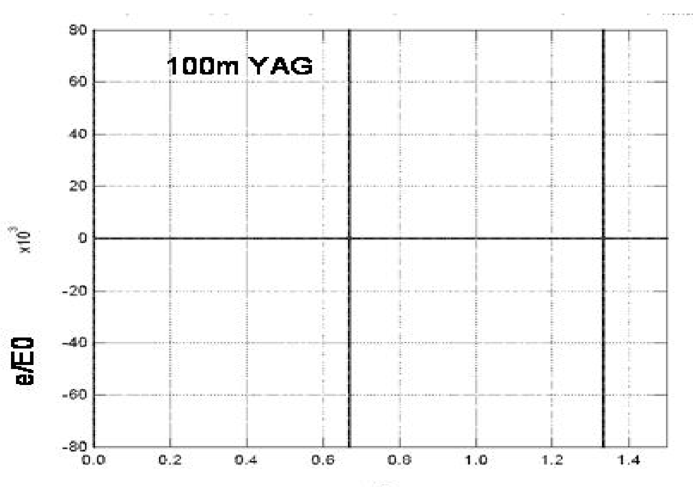


1.0
0.5
0.0
-0.5
-1.0
-1.5
optical period shown here
1.0039
1.0040
1.0041
1.0042
1.0043
1.0044
1.0045
ns

10 m YAG



8000
6000
4000
2000
0
-2000
-4000
-6000
-8000
e/E0
0
20
40
60
80
100
120
140
ns



100m YAG
80
60
40
20
0
-20
-40
-60
-80
x10^3
e/E0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
µs

$$P(\omega) = P_0 \sum_{-\infty}^{\infty} \left[\frac{(\Delta\omega/2)^2}{(n\omega_c)^2 + (\Delta\omega/2)^2} \right]^2 \delta(\omega - (\omega_0 + n\omega_c))$$

$P = IA$

$$\Rightarrow I(\omega) = I_0 \sum_{-\infty}^{\infty} \left[\frac{(\Delta\omega/2)^2}{(n\omega_c)^2 + (\Delta\omega/2)^2} \right]^2 \delta(\omega - (\omega_0 + n\omega_c))$$

$$\Rightarrow I_n = I_0 \left[\frac{(\Delta\omega/2)^2}{(n\omega_c)^2 + (\Delta\omega/2)^2} \right]^2$$

$$\Rightarrow E_n = E_0 \frac{(\Delta\omega/2)^2}{(n\omega_c)^2 + (\Delta\omega/2)^2}$$ where $E_n^2/2\eta_0 = I_n$

$$\Rightarrow e(t) = E_0 e^{j\omega_0 t} \sum_{-\infty}^{\infty} \frac{(\Delta\omega/2)^2}{(n\omega_c)^2 + (\Delta\omega/2)^2} e^{j\omega_n c t}$$

$$x = n\omega_c \Rightarrow dx = \omega_c \, dn$$

$$\Rightarrow e(t) = E_0 e^{j\omega_0 t} \frac{1}{\omega_c} \int_{-\infty}^{\infty} \frac{(\Delta\omega/2)^2}{x^2 + (\Delta\omega/2)^2} e^{jxt}$$

$$= E_0 e^{j\omega_0 t} \frac{1}{\omega_c} \int_{-\infty}^{\infty} \frac{(\Delta\omega/2)^2}{x^2 + (\Delta\omega/2)^2} \cos(xt)$$

$$= E_0 e^{j\omega_0 t} \frac{2}{\omega_c} \int_0^{\infty} \frac{(\Delta\omega/2)^2}{x^2 + (\Delta\omega/2)^2} \cos xt$$

Fourier Tranfourm $\frac{1}{x^2+b^2} \leftrightarrow \frac{\pi}{b} e^{-bx}$

OR Fourier COSINE transform $\frac{1}{x^2+b^2} \leftrightarrow \frac{\pi}{2b} e^{-bx}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(\Delta\omega/2)^2}{x^2+(\Delta\omega/2)^2} e^{jxt} = \left(\frac{\Delta\omega}{2}\right)^2 \pi \left(\frac{2}{\Delta\omega}\right) e^{-\frac{\Delta\omega}{2}t}$$

$$\Rightarrow e(t) = E_o e^{j\omega_o t} \frac{1}{\omega_c} \left(\frac{\Delta\omega}{2}\right)^2 \pi \left(\frac{2}{\Delta\omega}\right) e^{-\frac{\Delta\omega}{2}t}$$

$$= E_o e^{j\omega_o t} \frac{\pi}{2} \frac{\Delta\omega}{\omega_c} e^{-\frac{\Delta\omega}{2}t}$$

$$\Rightarrow \frac{P(t)}{P_o} = \left(\frac{\Delta\omega}{\omega_c}\right)^2 \frac{\pi^2}{4} e^{-\Delta\omega t}$$

$$\langle P \rangle = \frac{\omega_c}{2\pi}\int_{-\pi/\omega_c}^{\pi/\omega_c} \left(\frac{\Delta\omega}{\omega_c}\right)^2 \frac{\pi^2}{4} e^{-\Delta\omega t}\, dt$$

$$\rightarrow \frac{\omega_c}{2\pi}\left(\frac{\Delta\omega}{\omega_c}\right)^2 \frac{\pi^2}{4} \underbrace{\int_{-\infty}^{\infty} e^{-\Delta\omega t}\, dt}$$

$$= \frac{\omega_c}{2\pi}\left(\frac{\Delta\omega}{\omega_c}\right)^2 \frac{\pi^2}{4}\, 2\int_{0}^{\infty} e^{-\Delta\omega t}\, dt$$

$$= \frac{\omega_c}{2\pi}\left(\frac{\omega}{\omega_c}\right)^2 \frac{\pi^2}{4}\, \frac{2}{\Delta\omega} \underbrace{\int_{0}^{\infty} e^{-z}\, dz}_{1}$$

$$= \frac{\omega_c}{2\pi}\left(\frac{\omega}{\omega_c}\right)^2 \frac{\pi^2}{4}\, \frac{2}{\Delta\omega} = \frac{\pi}{4}\frac{\Delta\omega}{\omega_c} \checkmark$$

B) From above $\frac{P(0)}{P_0} = \frac{\pi^2}{4}\left(\frac{\Delta\omega}{\omega_c}\right)^2$

$$\Rightarrow \frac{P_{peak}}{P_0} = \pi \left(\frac{\Delta\omega}{\omega_c}\right) \langle P \rangle$$

c) $P(t) \sim e^{-\Delta\omega t}$

$$e^{-\Delta\omega t} = \frac{1}{2} \Rightarrow 2 = e^{\Delta\omega t}$$
$$\Rightarrow \ln 2 = \Delta\omega\, t$$
$$\Rightarrow t = \frac{\ln 2}{\Delta\omega}$$

(HWHM)

$$\text{FWHM} = \frac{2\ln 2}{\Delta\omega}$$